Math 3210-3
HW 6
Solutions

You were only required to turn in problems 1(b), 2, 3, 4(a), 5, 6, 7, 8, and 9. Each problem is worth one point. Parts (b) and (c) of 4 were extra credit. Note: Finish the whole assignment before attempting problem 4 as it is a more difficult problem.

Cardinality

1. Show that each of the following pairs of sets S and T are equinumerous by finding a specific bijection between them.

   (a) $S = [0, 1]$ and $T = [1, 3]$
   Let $f(x) = 2x + 1, 0 \leq x \leq 1$. Clearly this is a bijection from S to T.

   (b) $S = [0, 1]$ and $T = [0, 1)$
   We must shift infinitely many points in order to have our bijection, so define g as follows:

$$g(x) = \begin{cases} x & \text{if } x \neq \frac{1}{2^n} \text{ for any } n \in \mathbb{N} \cup \{0\} \\ \frac{1}{2^{n+1}} & \text{if } x = \frac{1}{2^n} \text{ for } n \in \mathbb{N} \cup \{0\} \end{cases}$$

   Now $1 \mapsto \frac{1}{2}$ and $\frac{1}{2} \mapsto \frac{1}{4}$, and so on. Clearly this is bijective.

   (c) $S = [0, 1)$ and $T = (0, 1)$
   We need a similar method as in part (b). Define h as follows:

$$h(x) = \begin{cases} x & \text{if } x \neq \frac{1}{3^n} \text{ for any } n \in \mathbb{N} \\ \frac{1}{3} & \text{if } x = 0 \\ \frac{1}{3^{n+1}} & \text{if } x = \frac{1}{3^n} \text{ for and } n \in \mathbb{N} \end{cases}$$

   (d) $S = (0, 1)$ and $T = (0, \infty)$
   Let $f(x) = \frac{1}{x} - 1$.

   (e) $S = (0, 1)$ and $T = \mathbb{R}$
   We will do this map in stages. We will take $(0, \frac{1}{2})$ to $(0, 1)$ and then $(0, 1)$ to $(-\infty, 0)$. Then we will take $(\frac{1}{2}, 1)$ to $(0, 1)$ to $(0, \infty)$. Finally we will send $\frac{1}{2}$ to 0. We can take $(0, \frac{1}{2})$ to $(0, 1)$ by the function $g(x) = 2x$. Then we can compose this with the function from part (d) to get $(0, \frac{1}{2})$ to $(0, \infty)$. Next, multiply this function by a negative to get it to be in $(-\infty, 0$. The second function is similar. We take $(\frac{1}{2}, 1)$ to $(0, 1)$ by the function $h(x) = 2(x - \frac{1}{2}) = 2x - 1$. Compose this with the map from (d) to get our desired function. In summary, we have the following bijection:

$$f(x) = \begin{cases} -\left(\frac{1}{2x} - 1\right) & \text{if } 0 < x < \frac{1}{2} \\ 0 & \text{if } x = \frac{1}{2} \\ \frac{1}{2x-1} - 1 & \text{if } \frac{1}{2} < x < 1 \end{cases}$$

2. (a) Suppose that $m < n$. Prove that the intervals $(0, 1)$ and (m, n) are equinumerous by finding a specific bijection between them.
   Proof: We can use our method from part (a) of the previous problem. We simply want a straight line with $0 \mapsto m$ and $1 \mapsto n$. So our points for our line are $(0, m), (1, n)$. The slope of this line is $\frac{n-m}{1-0} = n - m$. The y-intercept is $(0, m)$, so the equation of our line is $f(x) = (n - m)x + m$.

   □

   (b) Use part (a) to prove that any two open intervals are equinumerous.
   Proof: Let (a, b) and (c, d) be open intervals with $a < b$ and $c < d$. By the method of part (a), we can find the equation of the straight line from (a, c) to (b, d). The slope of this line is $\dfrac{d - c}{b - a}$, so the bijection is $f(x) = \dfrac{d - c}{b - a}(x - a) + c$.

□

3. ♣ Prove: If $(S \setminus T) \sim (T \setminus S)$, then $S \sim T$.

Proof: Let $f : S \setminus T \to T \setminus S$ be a bijection. Define $g : S \to T$ by

$$g(x) = \begin{cases} f(x) & \text{if } x \in S \setminus T \\ x & \text{if } x \in S \cap T \end{cases}$$

We need to show that g is bijective. Suppose $g(x) = g(y)$. We have three cases to consider. First, suppose $x, y \in S \setminus T$. Then $g(x) = f(x) = f(y) = g(y)$, so $x = y$ since f is injective. Second, suppose $x, y \in S \cap T$. Then $g(x) = x = y = g(y)$, so $x = y$ since the identity function is injective. Finally, suppose $x \in S \setminus T$ and $y \in S \cap T$. Then $g(x) = f(x) = y = g(y)$ implies that $f(x) \in S \cap T$, a contradiction since $f(x) \in T \setminus S$. Therefore g is injective.

Now let $t \in T$. If $t \in S \cap T$, then $t \in S$ and $g(t) = t$. If $t \in S \setminus T$, then $\exists x \in S$ such that $f(x) = t$ since f is bijective. Therefore g is surjective. And we can conclude that g is in fact bijective. Therefore $S \sim T$.

□

4. A real number is said to be algebraic if it is a root of a polynomial equation

$$a_n x^n + \cdots + a_1 x + a_0 = 0$$

with integer coefficients. Note that the algebraic numbers include the rationals and all roots of rationals (such as $\sqrt{2}$, $\sqrt[3]{5}$, etc.). If a number is not algebraic, it is called transendental.

(a) Show that the set of polynomials with integer coefficients is countable. (Hint: use the fact that the countable union of countable sets is countable.)

We will begin this problem by proving a lemma.

Lemma 1
Let $\mathbb{Z}^i = \mathbb{Z} \times \mathbb{Z} \times \cdots \times \mathbb{Z}$ where there are i factors in the product. Then $\mathbb{Z}^i$ is countable.

Proof: We will prove this by induction (something we will learn in the next section). Let $i = 1$. Then $\mathbb{Z}^1 = \mathbb{Z}$. Let $f : \mathbb{Z} \to \mathbb{N}$ be given by

$$f(x) = \begin{cases} 2x & x \geq 0 \\ -2x - 1 & x < 0 \end{cases}$$

By theorem 16, we must only show that $f(x)$ is injective for $\mathbb{Z}$ to be countable. Suppose $f(x) = f(y)$. Then there are three cases to consider. Case 1: $x, y \geq 0$. Then $f(x) = 2x = 2y = f(y)$, which implies that $x = y$. Case 2: $x, y < 0$, so $f(x) = -2x - 1 = -2y - 1 = f(y)$, so $x = y$. And Case 3: $x \geq 0$ and $y < 0$. Then $f(x) = 2x = -2y - 1 = f(y)$ which implies that $x = -y - \frac{1}{2} \notin \mathbb{Z}$, so we have a contradiction. Therefore, f is injective, and $\mathbb{Z}$ is countable.

Now suppose that $\mathbb{Z}^k$ is countable for some $k \in \mathbb{N}$. Now we need to show that $\mathbb{Z}^{k+1}$ is countable. But $\mathbb{Z}^{k+1} = \mathbb{Z}^k \times \mathbb{Z}$. Since $\mathbb{Z}^k$ and $\mathbb{Z}$ are countable, by an exercise we did in class, $\mathbb{Z}^{k+1}$ is countable. Therefore, by induction, we know that $\mathbb{Z}^i$ is countable for all $i \in \mathbb{N}$.

□

Now we will prove the homework problem.

Proof: Let $\mathcal{A}$ = the set of all polynomials with integer coefficients. Let A_i be the set of all polynomials of degree i with integer coefficients. Then $\mathcal{A} = \cup A_i$, which is a countable union. Let $h_i : A_i \to \mathbb{Z}^{i+1}$ be given by $h_i(a_i x^i + a_{i-1} x^{i-1} + \cdots + a_1 x + a_0) = (a_i, a_{i-1}, \ldots, a_1, a_0)$. Clearly h_i is injective. By the lemma we just proved, $\mathbb{Z}^{i+1}$ is countable, so $\exists f_{i+1} : \mathbb{Z}^{i+1} \to \mathbb{N}$ which is bijective. Define $g_i : A_i \to \mathbb{N}$ by $g_i(p) = f_{i+1} \circ h_i(p)$ for any $p \in A_i$. The composition of injective functions is injective, as we proved in class, so A_i is countable for all $i \in \mathbb{N}$. Finally, $\mathcal{A}$ is a countable union of countable sets, so it is countable, by an exercise from class.

□

(b) Show that the set of algebraic numbers is countable.

Proof: Let A_i be defined as above. Note that A_i is countable for all $i \in \mathbb{N}$, so we can label all of the polynomials in A_i as $p_1, p_2, \ldots$. For and $k \in \mathbb{N}$, $p_k \in A_i$ has at most i roots. (Note that we don't need finite roots, we just need countable roots, but finite is still countable.) Let r_1 = roots of p_1, r_2 = roots of p_2, and so on. Now let $R_i = \cup r_j$ which are all of the roots of polynomials of degree i. R_i is countable since it is the countable union of countable sets. Also note that there is such an R_i for each A_i. Let $\mathcal{R} = \cup R_i$. Then $\mathcal{R}$ is the set of all of the algebraic numbers, and it is countable since it is the countable union of countable sets.

$\square$

(c) Are there more algebraic numbers or transcendental numbers?

Since there are uncountable many real numbers, and $\mathbb{R}$ = algebraic numbers $\cup$ transcendental numbers, there must be more transcendental numbers then algebraic numbers.