

**Math 3210-3**  
**HW 5**  
Solutions

This homework is worth 3 points.

## Functions

1. ♣ Suppose that  $f : A \rightarrow B, g : B \rightarrow C$ , and  $h : C \rightarrow D$ . Prove that  $h \circ (g \circ f) = (h \circ g) \circ f$ .

*Proof:* By the definition of composite functions we have

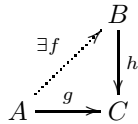
$$\begin{aligned} h \circ (g \circ f) &= \{(a, d) | \exists c \in C \ni (a, c) \in g \circ f, (c, d) \in h\} \\ &= \{(a, d) | \exists c \in C, \exists b \in B \ni (a, b) \in f, (b, c) \in g, (c, d) \in h\} \\ &= \{(a, d) | \exists b \in B \ni (a, b) \in f, (b, d) \in h \circ g\} \\ &= (h \circ g) \circ f \end{aligned}$$

□

2. Find an example of functions  $f : A \rightarrow B$  and  $g : B \rightarrow C$  such that  $f$  and  $g \circ f$  are both injective but  $g$  is not injective.

Let  $f : \mathbb{N} \rightarrow \{2, 3, \dots\}$  be given by  $f(n) = 2n$ , and let  $g : \{2, 3, \dots\} \rightarrow \mathbb{N}$  be given by  $g(n) = \lfloor \frac{n}{2} \rfloor$ . Then clearly  $f$  is injective and  $g$  is not. Also,  $g \circ f(n) = \lfloor \frac{2n}{2} \rfloor = \lfloor n \rfloor = n$  which is also injective.

3. ♣ Suppose that  $g : A \rightarrow C$  and  $h : B \rightarrow C$ . Prove that if  $h$  is bijective then there exists a function  $f : A \rightarrow B$  such that  $g = h \circ f$ . Hint: Draw a picture.



*Proof:* We will construct  $f$ . Let  $a \in A$ . Since  $h$  is bijective, there exists a unique  $b \in B$  such that  $g(a) = h(b)$ . Define  $f(a) = b$ . Clearly  $h \circ f(a) = h(b) = g(a)$ , so  $g = h \circ f$ . We must only show  $f$  is a function. Suppose  $f(a) = b_1$  and  $f(a) = b_2$ . Then by the way  $f$  is defined,  $g(a) = h(b_1)$  and  $g(a) = h(b_2)$ , a contradiction. Therefore  $f$  is our desired function.

□