Math 3210-3
HW 4
Solutions

You were only required to turn in the following problems: 1(e), 2(a), 3, 4, 5(b), 6, 7(b), and 8(b). Each problem is worth 1 point.

Relations

1. Determine which of the three properties (reflexive, symmetric and transitive) apply to each relation. (You can write simply "R," "S," or "T".)

   (a) Let R be the relation on $\mathbb{N}$ given by xRy iff x divides y. **R, T**

   (b) Let X be a set and let R be the relation "$' \subseteq$"' defined on subsets of X. **R, T**

   (c) Let R be the relation on the real numbers given by xRy iff $x - y$ is rational. **R, S, T**

   (d) Let R be the relation on the real numbers given by xRy iff $x - y$ is irrational. **S**

   (e) Let R be the relation on the real numbers given by xRy iff $(x - y)^2 < 0$. **S, T**

   (f) Let R be the relation on the real numbers given by xRy iff $|x - y| \leq 2$. **R, S**

2. Find examples of relations with the following properties.

   (a) Reflexive, but not symmetric and not transitive.
   Let R be the relation on $\mathbb{R}$ given by xRy if and only if $x < y + 1$.

   (b) Symmetric, but not reflexive and not transitive.
   Let R be the relation on $\mathbb{R}$ given by xRy iff $x - y$ is irrational.

   (c) Transitive, but not reflexive and not symmetric.
   Let R be the relation on $\mathbb{R}$ given by xRy iff $x < y$.

   (d) Reflexive and symmetric, but not transitive.
   Let R be the relation on $\mathbb{R}$ given by xRy iff $|x - y| \leq 2$.

   (e) Reflexive and transitive, but not symmetric.
   Let R be the relation on $\mathbb{N}$ given by xRy iff x divides y.

   (f) Symmetric and transitive, but not reflexive.
   Let R be the relation on the real numbers given by xRy iff $(x - y)^2 < 0$.

3. ♣ Let S be the Cartesian coordinate plane $\mathbb{R} \times \mathbb{R}$ and define a relation R on S by $(a, b)R(c, d)$ iff $a = c$. Verify that R is an equivalence relation and describe a typical equivalence class $E_{(a,b)}$.

   Proof: To show R is an equivalence relation, we need to show it is reflexive, symmetric, and transitive. Clearly we have that R is reflexive since $(a, b)R(a, b)$ iff $a = a$ which is always true. Now suppose $(a, b)R(c, d)$. Then $a = c$ and $(c, d)R(a, b)$ is true. Hence R is symmetric. Finally suppose $(a, b)R(c, d)$ and $(c, d)R(e, f)$. Then $a = c$ and $c = e$, so $a = e$, and we have that $(a, b)R(e, f)$ and R is transitive.

   □

   A typical equivalence class would be $E_{(a,b)} = \{(a, x) | x \in \mathbb{R}\}$ which is a vertical line for each value of a.

4. Let S be the set $\mathbb{Z}$ of all integers and let $R = \{(m, n) \in \mathbb{Z} \times \mathbb{Z} : m - n \text{ is even}\}$. Verify that R is an equivalence relation and describe the equivalence class E_5. How many distinct equivalence classes are there?

   Proof: Once again we need to show R is reflexive, symmetric, and transitive. Then mRm iff $m - m$ is even, but $m - m = 0 = 2(0)$ which is even, so mRm. If mRn, then $m - n$ is even, and we can write $m - n = 2k$ for some $k \in \mathbb{Z}$. Then $n - m = -(m - n) = -(2k) = 2(-k)$ which is also even, so nRm. Now suppose that mRn and nRr. Then $m - n = 2k$ and $n - r = 2l$ for $k, l \in \mathbb{Z}$. Then $m - r = m - n + n - r = (m - n) + (n - r) = 2k + 2l = 2(k + l)$ which is even, so mRr. Therefore R is an equivalence relation.

□

The equivalence class $E_5 = \{(5,n) | 5-n \text{ is even}\} = \{(5,n) | n \text{ is odd}\} = \{(a,b) | a, b \text{ are odd}\}$. Therefore there are two distinct equivalence classes, E_{odd} and E_{even}.

Functions

5. Find the range of each function $f : \mathbb{R} \to \mathbb{R}$.

   (a) $f(x) = x^2 + 1$.
   The graph of this function is a parabola with vertex (0,1) which opens up. So the range is $[1, \infty)$.

   (b) $f(x) = (x+3)^2 - 5$.
   The graph of this function is a parabola with vertex (-3,-5) which opens up. So the range is $[-5, \infty)$.

   (c) $f(x) = x^2 + 4x + 1$.
   The graph of this function is a parabola. We can complete the square to find the vertex which is at (-2,-3), so the range is $[-3, \infty)$.

   (d) $f(x) = 2\cos 3x$.
   The range of a cosine function is $[-1, 1]$, but since we have altered the amplitude of this function by multiplying by 2, the range of this function is $[-2, 2]$.

6. Let S be the set of all circles in the plane, and let T be the set of all circles in the plane that are centered at the origin. Then $T \subset S$.

   (a) Define $f : S \to [0, \infty)$ by $f(C) =$ the area enclosed by C, for all $C \in S$. Is f injective? Is f surjective?
   f is not injective. To see this, let C_1 be the circle of area 1 centered at the origin, and let C_2 be the circle of area 1 centered at $(1,1)$. Then $f(C_1) = 1 = f(C_2)$ but the circles are distinct. On the other hand, f is surjective. To see this, choose any $r \in \mathbb{R}$. Let C be the circle with area r centered at the origin. Then $f(C) = r$.

   (b) Define $g : T \to [0, \infty)$ by $g(C) =$ the area enclosed by C, for all $C \in T$. Is g injective? Is g surjective?
   In this case g is injective and surjective. Two circles with the same area centered at the origin have to be the same circle, so g is injective. And by the same argument as above, the function is surjective.

7. ♣ In each part, find a function $f : \mathbb{N} \to \mathbb{N}$ that has the desired properties.

   (a) surjective, but not injective.
   Let $f(n) = [[1 + \frac{n}{3}]]$. Then $f(1) = 1$ and $f(2) = 1$, but for any $m \in \mathbb{N}$, $f(3(m-1)) = m$, so f is surjective, but not injective.

   (b) injective, but not surjective.
   Let $f(n) = n^2$. Clearly f is injective, but not surjective. For instance, consider $2 \in \mathbb{N}$. There is no element in $\mathbb{N}$ such that $n^2 = 2$.

   (c) neither surjective nor injective.
   Let $f(n) = 5$. Then clearly f is not surjective nor injective.

   (d) bijective.
   Let $f(n) = n$. Clearly f is injective and surjective.

8. ♣ Find examples to show that equality does not hold in the following theorem: Suppose that $f : A \to B$. Let C, C_1 and C_2 be subsets of A and let D, D_1, and D_2 be subsets of B. Then the following hold:

   (a) $C \subseteq f^{-1}[f(C)]$.
   Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2$. Let $C = [0, \infty)$. Then $f^{-1}(f(C)) = f^{-1}(f([0, \infty))) = f^{-1}([0, \infty)) = \mathbb{R}$.

(b) $f[f^{-1}(D)] \subseteq D$.
Let $f : \mathbb{N} \to \mathbb{N}$ be given by $f(n) = n + 1$. Let $D = \mathbb{N}$. Then $f[f^{-1}(D)] = f(\mathbb{N}) = \{2, 3, \ldots\}$.

(c) $f(C_1 \cap C_2) \subseteq f(C_1) \cap f(C_2)$.
Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2$, and let $C_1 = (-2, -1)$ and $C_2 = (1, 2)$. Then $f(C_1) = (1, 4), f(C_2) = (1, 4), C_1 \cap C_2 = \emptyset$, so $f(C_1 \cap C_2) = \emptyset$, but $f(C_1) \cap f(C_2) = (1, 4)$.