Math 3210-3
HW 4
Due Tuesday, September 4, 2007

You are only required to turn in the following problems: 1(e), 2(a), 3, 4, 5(b), 6, 7, 8(b), 9(b), 10, and 11. Each problem is worth 1 point.

Relations

1. Determine which of the three properties (reflexive, symmetric and transitive) apply to each relation. (You can write simply "R," "S," or "T".)
   (a) Let R be the relation on $\mathbb{N}$ given by xRy iff x divides y.
   (b) Let X be a set and let R be the relation " $\subseteq$ " defined on subsets of X.
   (c) Let R be the relation on the real numbers given by xRy iff $x - y$ is rational.
   (d) Let R be the relation on the real numbers given by xRy iff $x - y$ is irrational.
   (e) Let R be the relation on the real numbers given by xRy iff $(x - y)^2 < 0$.
   (f) Let R be the relation on the real numbers given by xRy iff $|x - y| \leq 2$.
2. Find examples of relations with the following properties.
   (a) Reflexive, but not symmetric and not transitive.
   (b) Symmetric, but not reflexive and not transitive.
   (c) Transitive, but not reflexive and not symmetric.
   (d) Reflexive and symmetric, but not transitive.
   (e) Reflexive and transitive, but not symmetric.
   (f) Symmetric and transitive, but not reflexive.
3. ♣ Let S be the Cartesian coordinate plane $\mathbb{R} \times \mathbb{R}$ and define a relation R on S by $(a, b)R(c, d)$ iff $a = c$. Verify that R is an equivalence relation and describe a typical equivalence class $E_{(a,b)}$.
4. Let S be the set $\mathbb{Z}$ of all integers and let $R = \{(m, n) \in \mathbb{Z} \times \mathbb{Z} : m - n \text{ is even}\}$. Verify that R is an equivalence relation and describe the equivalence class E_5. How many distinct equivalence classes are there?

Functions

5. Find the range of each function $f : \mathbb{R} \to \mathbb{R}$.
   (a) $f(x) = x^2 + 1$.
   (b) $f(x) = (x + 3)^2 - 5$.
   (c) $f(x) = x^2 + 4x + 1$.
   (d) $f(x) = 2 \cos 3x$.
6. Let S be the set of all circles in the plane, and let T be the set of all circles in the plane that are centered at the origin. Then $T \subset S$.
   (a) Define $f : S \to [0, \infty)$ by $f(C) =$ the area enclosed by C, for all $C \in S$. Is f injective? Is f surjective?
   (b) Define $g : T \to [0, \infty)$ by $g(C) =$ the area enclosed by C, for all $C \in T$. Is g injective? Is g surjective?
7. ♣ Suppose that $f : A \to B, g : B \to C$, and $h : C \to D$. Prove that $h \circ (g \circ f) = (h \circ g) \circ f$.

8. ♣ In each part, find a function $f : \mathbb{N} \to \mathbb{N}$ that has the desired properties.

   (a) surjective, but not injective.
   (b) injective, but not surjective.
   (c) neither surjective nor injective.
   (d) bijective.

9. ♣ Find examples to show that equality does not hold in the following theorem: Suppose that $f : A \to B$. Let C, C_1 and C_2 be subsets of A and let D, D_1, and D_2 be subsets of B. Then the following hold:

   (a) $C \subseteq f^{-1}[f(C)]$.
   (b) $f[f^{-1}(D)] \subseteq D$.
   (c) $f(C_1 \cap C_2) \subseteq f(C_1) \cap f(C_2)$.

10. Find an example of functions $f : A \to B$ and $g : B \to C$ such that f and $g \circ f$ are both injective but g is not injective.

11. ♣ Suppose that $g : A \to C$ and $h : B \to C$. Prove that if h is bijective then there exists a function $f : A \to B$ such that $g = h \circ f$. Hint: Draw a picture.