

**Math 3210-3**  
**HW 3**  
Solutions

There are 7 points possible on this assignment. Only part (b) of problem 3 will be graded.

## Set Operations

1. Prove:  $(A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)$ .

*Proof* We will first show that  $(A \setminus B) \cup (B \setminus A) \subseteq (A \cup B) \setminus (A \cap B)$ . Let  $x \in (A \setminus B) \cup (B \setminus A)$ . Then either  $x \in A \setminus B$  or  $x \in B \setminus A$ . If  $x \in A \setminus B$ , then  $x \in A$  and  $x \notin B$ , so  $x \notin A \cap B$ , and  $x \in A \cup B$ , so  $x \in (A \cup B) \setminus (A \cap B)$ . On the other hand, if  $x \in B \setminus A$ , then  $x \in B$  and  $x \notin A$ . By the same argument,  $x \notin A \cap B$ , so  $x \in (A \cup B) \setminus (A \cap B)$ . Hence  $(A \setminus B) \cup (B \setminus A) \subseteq (A \cup B) \setminus (A \cap B)$ .

Conversely, we will show  $(A \cup B) \setminus (A \cap B) \subseteq (A \setminus B) \cup (B \setminus A)$ . Let  $y \in (A \cup B) \setminus (A \cap B)$ . Then  $y \in A$  or  $y \in B$ , but  $y \notin (A \cap B)$ . So if  $y \in A$  and  $y \notin (A \cap B)$ , then  $y \in A \setminus B$ . If  $y \in B$  and  $y \notin (A \cap B)$ , then  $y \in B \setminus A$ . Hence  $(A \cup B) \setminus (A \cap B) \subseteq (A \setminus B) \cup (B \setminus A)$ .

□

2. ♣ Prove:  $A \cap B = A \setminus (A \setminus B)$ .

*Proof* First we will show  $A \cap B \subseteq A \setminus (A \setminus B)$ . Let  $x \in A \cap B$ . Then  $x \in A$  and  $x \in B \implies x \notin A \setminus B \implies x \in A \setminus (A \setminus B)$ . Hence  $A \cap B \subseteq A \setminus (A \setminus B)$ .

Conversely, we now want to show  $A \setminus (A \setminus B) \subseteq A \cap B$ . Let  $y \in A \setminus (A \setminus B)$ . Then  $y \in A$  and  $y \notin A \setminus B \implies y \in A$  and  $y \in B \implies y \in A \cap B$ . Therefore  $A \cap B = A \setminus (A \setminus B)$ .

□

3. Let  $\{A_j : j \in J\}$  be an indexed family of sets and let  $B$  be a set. Prove the following generalizations of theorem 6 from class.

(a)  $B \cup \left[ \bigcap_{j \in J} A_j \right] = \bigcap_{j \in J} (B \cup A_j)$ .

*Proof* Let  $x \in B \cup \left[ \bigcap_{j \in J} A_j \right]$ . Then either  $x \in B$  or  $x \in A_j$  for all  $j \in J$ . If  $x \in B$ , then  $x \in B \cup A_j$  for all  $j \in J$  which implies that  $x \in \bigcap_{j \in J} (B \cup A_j)$ . On the other hand, if  $x \in A_j \forall j \in J$ , then  $x \in B \cup A_j \forall j \in J$  which implies that  $x \in \bigcap_{j \in J} (B \cup A_j)$ . Hence  $B \cup \left[ \bigcap_{j \in J} A_j \right] \subseteq \bigcap_{j \in J} (B \cup A_j)$ . For the reverse inclusion, let  $y \in \bigcap_{j \in J} (B \cup A_j)$ . Then  $y \in B \cup A_j \forall j \in J$  which implies that  $y \in B$  or  $y \in A_j$  for all  $j \in J$ . Hence  $y \in B \cup \left[ \bigcap_{j \in J} A_j \right]$ . Therefore  $B \cup \left[ \bigcap_{j \in J} A_j \right] = \bigcap_{j \in J} (B \cup A_j)$ .

□

(b)  $B \cap \left[ \bigcup_{j \in J} A_j \right] = \bigcup_{j \in J} (B \cap A_j)$ .

*Proof* Let  $x \in B \cap \left[ \bigcup_{j \in J} A_j \right]$ . Then  $x \in B$  and  $x \in \bigcup_{j \in J} A_j$ , so  $x \in B$  and  $\exists j$  such that  $x \in A_j$ . Thus for some  $j \in J$ ,  $x \in B \cap A_j$ , so  $x \in \bigcup_{j \in J} (B \cap A_j)$ .

For the reverse inclusion, let  $y \in \bigcup_{j \in J} (B \cap A_j)$ . Then  $\exists j \in J$  such that  $y \in B \cap A_j$ , which implies that  $y \in B$  and  $y \in A_j$ . Hence  $y \in B \cap \left[ \bigcup_{j \in J} A_j \right]$ .

□

(c)  $B \setminus \left[ \bigcup_{j \in J} A_j \right] = \bigcap_{j \in J} (B \setminus A_j)$ .

*Proof* Let  $x \in B \setminus \left[ \bigcup_{j \in J} A_j \right]$ . Then  $x \in B$  and  $\forall j \in J$ ,  $x \notin A_j$ . So  $x \in B \setminus A_j \forall j \in J$ . Thus  $x \in \bigcap_{j \in J} (B \setminus A_j)$ .

For the reverse inclusion, let  $y \in \bigcap_{j \in J} (B \setminus A_j)$ . So  $\forall j$ ,  $y \in B$  and  $y \notin A_j$ . Hence  $y \in B \setminus \left[ \bigcup_{j \in J} A_j \right]$ .

Therefore  $B \setminus \left[ \bigcup_{j \in J} A_j \right] = \bigcap_{j \in J} (B \setminus A_j)$ .

□

$$(d) B \setminus \left[ \bigcap_{j \in J} A_j \right] = \bigcup_{j \in J} (B \setminus A_j).$$

*Proof* Let  $x \in B \setminus \left[ \bigcap_{j \in J} A_j \right]$ . Then  $x \in B$  and  $\exists j$  such that  $x \notin A_j$ . For that  $j$ , we have  $x \in B \setminus A_j$ . Hence  $x \in \bigcup_{j \in J} (B \setminus A_j)$ .

For the reverse inclusion, suppose  $y \notin B \setminus \left[ \bigcap_{j \in J} A_j \right]$ . Then  $y \in \bigcap_{j \in J} A_j$ , so for all  $j \in J$ ,  $y \in A_j$ . So  $y \notin B \setminus A_j$  for all  $j$ . Therefore  $y \notin \bigcup_{j \in J} (B \setminus A_j)$ .

□

## Relations

4. ♣ Prove or give a counterexample:  $A \times B = B \times A$ .

Consider the following counterexample. Let  $A = \{1\}$ ,  $B = \{x \in \mathbb{R} \mid 5 \leq x \leq 6\}$ . Then  $A \times B = \{(1, x) \mid 5 \leq x \leq 6\}$  which is a vertical line segment in the plane, and  $B \times A = \{(x, 1) \mid 5 \leq x \leq 6\}$  which is a horizontal line segment in the plane.

5. Prove or give a counterexample:

$$(a) (A \cup B) \times C = (A \times C) \cup (B \times C).$$

*Proof* Let  $(x, y) \in (A \cup B) \times C$ . Then  $x \in A \cup B$  and  $y \in C$ . So  $x \in A$  and  $y \in C$  or  $x \in B$  and  $y \in C$ . In the first case,  $(x, y) \in A \times C$ , and in the second case,  $(x, y) \in B \times C$ . Thus  $(x, y) \in (A \times C) \cup (B \times C)$ .

For the reverse inclusion, let  $(z, w) \in (A \times C) \cup (B \times C)$ . Then  $(z, w) \in A \times C$  or  $(z, w) \in B \times C$ . So  $z \in A$  or  $z \in B$ , and  $w \in C$ . Hence  $(z, w) \in (A \cup B) \times C$ .

□

$$(b) (A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D).$$

*Proof* We have that  $(x, y) \in (A \times B) \cap (C \times D)$  if and only if  $(x, y) \in A \times B$  and  $(x, y) \in C \times D$  if and only if  $x \in A$  and  $x \in C$  and  $y \in B$  and  $y \in D$  if and only if  $(x, y) \in (A \cap C) \times (B \cap D)$ .

□

$$(c) (A \times B) \cup (C \times D) = (A \cup C) \times (B \cup D).$$

We will provide a counterexample. Let  $A, B, C$  and  $D$  be pairwise disjoint sets. Consider  $(a, d)$  where  $a \in A$  and  $d \in D$ . Then  $(a, d) \in (A \cup C) \times (B \cup D)$ , but  $(a, d) \notin A \times B$  and  $(a, d) \notin C \times D$ .