Math 3210-3

HW 2

Solutions

There are 10 points possible on this assignment plus 2 extra credit points. Each problem is worth one point except problem 3 and the extra credit problem which are worth two points each. The problems with ♣ are problems which we could present in our problem sessions.

Techniques of Proof I

1. Write the contrapositive of each implication.

   (a) If all roses are red, then all violets are blue.

   (b) G is normal if G is not regular.

   (c) If K is closed and bounded, then K is compact.

   Solution 1

   (a) If all violets are not blue, then all roses are not red.

   (b) If G is not normal, then G is regular.

   (c) If K is not compact, then K is not closed or not bounded.

2. Write the converse of each implication in Exercise 1.

   Solution 2

   (a) If all violets are blue, then all roses are red.

   (b) If G is normal, then G is not regular.

   (c) If K is compact, then K is closed and bounded.

3. Provide a counterexample for each statement.

   (a) For every real number x, if $x^2 > 4$, then $x > 2$.

   (b) For every positive integer n, $n^2 + n + 41$ is prime.

   (c) Every triangle is right triangle.

   (d) No integer greater then 100 is prime.

   (e) Every prime number is odd.

   (f) For every positive integer n, $3n$ is divisible by 6.

   (g) No rational number satisfies the equation $x^3 + (x-1)^2 = x^2 + 1$.

   (h) No rational number satisfies the equation $x^4 + (1/x) - \sqrt{x+1} = 0$.

   Solution 3

   (a) Let $x = -3$. Then x is a real number, $x^2 = 9 > 4$, but $-3 < 2$.

   (b) Let $n = 41$. Clearly 41 is a positive integer, but $41^2 + 41 + 41 = 41(41 + 1 + 1) = 41(43)$ which is not prime.

   (c) Let T be any triangle with angles 10 degrees, 20 degrees, and 150 degrees. This is a triangle because the sum of the angles is 180 degrees, but none of the angles is 90 degrees, so it is not a right triangle.

   (d) Let $n = 101$. Clearly $101 > 100$, and it is easy to check that it is prime. Or let $n = 2^{24,036,583} - 1$, the 41^{st} known Mersenne Prime. See http://www.mersenne.org/prime.htm

   (e) Let $p = 2$. 2 is prime, but it is even.

   (f) Let $n = 1$. Then $3n = 3$ which is not divisible by 6.

(g) Let $x = 0$. We see that $0^3 + (0-1)^2 = 1 = 0^2 + 1$.

(h) Let $x = -1$. It is easy to see that this rational number satisfies the equation.

4. ♣ Let f be the function given by $f(x) = 3x - 5$. Use the contrapositive implication to prove: If $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$.

Solution 4

Proof We will prove the contrapositive of this statement which is: If $f(x_1) = f(x_2)$ then $x_1 = x_2$. So suppose $f(x_1) = f(x_2)$. This means $3x_1 - 5 = 3x_2 - 5$. Add 5 to both sides and then divide by 3 to get $x_1 = x_2$.

□

5. Use the contrapositive implication to prove: If n^2 is an even number, then n is an even number. (Hint: A number is odd iff it can be written as $2k + 1$ for some integer k.)

Solution 5

Proof We will prove the contrapositive of the statement which is: If n is an odd number, then n^2 is an odd number. Since n is odd, we can write $n = 2k + 1$ where $k \in \mathbb{Z}$. Then $n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$. Therefore n^2 is odd.

□

Techniques of Proof II

6. Prove: There exists an integer n such that $n^2 + \dfrac{3n}{2} = 1$. Is this integer unique?

Proof Let $n = -2$. Then $n \in \mathbb{N}$ and we see that $(-2)^2 + \frac{3(-2)}{2} = 1$. We know from the fundamental theorem of algebra that there are exactly two solutions to this equation in the reals. If we solve the equation, we see that $n = \frac{1}{2}$ also satisfies the equation, but $\frac{1}{2} \notin \mathbb{N}$. Therefore $n = -2$ is the unique integer solution.

□

7. ♣ Prove: If x is a real number, then $|x - 2| \leq 3$ implies that $-1 \leq x \leq 5$.

Proof We will prove the contrapositive statement, so we have two cases. First suppose $x < -1$. Then $x - 2 < -1 - 2 = -3 \implies |x - 2| = -(x - 2) > 3$, as desired. On the other hand, if $x > 5$ we see $x - 2 > 3 \implies |x - 2| = x - 2 > 3$. Therefore if x is a real number, then $|x - 2| \leq 3$ implies $-1 \leq x \leq 5$.

□

Proof We can solve the equation $|x - 2| \leq 3$ for x. There are two cases to consider: if $x - 2 \geq 0$ and if $x - 2 < 0$. If $x - 2 \geq 0$, then $x \geq 2$, and we know $|x - 2| = x - 2$. So we will solve the equation $x - 2 \leq 3 \implies x \leq 5$ Hence in this case $2 \leq x \leq 5$. On the other hand, if $x - 2 < 0$, then $x < 2$, and we know $|x - 2| = -(x - 2) = -x + 2$. So $|x - 2| \leq 3 \implies -x + 2 \leq 3 \implies -x \leq 1 \implies x \geq -1$. In this case our solution is $-1 \leq x < 2$. Combining these two inequalities, we get $-1 \leq x \leq 5$.

□

8. ♣ Prove: If $\dfrac{x}{x-1} \leq 2$, then $x < 1$ or $x \geq 2$.

Proof We will show that if $\dfrac{x}{x-1} \leq 2$ and $x > 1$, then $x \geq 2$. If $x > 1$ then $x - 1 > 0$, so when we multiply the equation $\dfrac{x}{x-1} \leq 2$ by $x - 1$ the inequality stays the same. Thus we have $x \leq 2(x - 1) \implies x \leq 2x - 2 \implies 2 \leq x$, as desired.

□

Proof There are two cases to consider. First, suppose $x - 1 > 0$. Then $x > 1$ and when we solve the equation we get $\frac{x}{x-1} \leq 2 \Longrightarrow x \leq 2(x-1) \Longrightarrow x \leq 2x - 2 \Longrightarrow 2 \leq x$. So in this case our solution is $x > 1$ and $2 \leq x$ which can be combined to the statement that $x \geq 2$.

On the other hand, if $x - 1 < 0$ then $x < 1$ and when we solve the equation we get $\frac{x}{x-1} \leq 2 \Longrightarrow x \geq 2(x-1) \Longrightarrow x \geq 2x - 2 \Longrightarrow 2 \geq x$ and $x < 1$ which gives us $x < 1$. Therefore $x \geq 2$ or $x < 1$.

□

9. ♣ Prove or give a counterexample: For every positive integer n, $n^2 + 3n + 8$ is even.

   Proof We will consider the two cases when n is even and when n is odd. If n is even, we can write $n = 2k$ for $k \in \mathbb{N}$. Then $n^2 + 3n + 8 = (2k)^2 + 3(2k) + 8 = 4k^2 + 6k + 8 = 2(2k^2 + 3k + 4)$ which is even. If n is odd, we can write $n = 2k + 1$ for $k \in \mathbb{N}$. Then we have $n^2 + 3n + 8 = (2k+1)^2 + 3(2k+1) + 8 = 4k^2 + 4k + 1 + 6k + 3 + 8 = 4k^2 + 10k + 12 = 2(2k^2 + 5k + 6)$ which is also even.

□

The following problem is for **Extra Credit**. I typically do not give extra credit problems, but I really liked this problem, and I think students should at least think about it even if they decide not to complete the problem.

10. Consider the following sentences:

    (a) The nucleus of a carbon atom consists of protons and neutrons.

    (b) Jesus Christ rose from the dead and is alive today.

    (c) Every differentiable function is continuous.

    Each of these sentences has been affirmed by some people at some time as being "true."' Write an essay on the nature of truth, comparing and contrasting its meaning in these (and possibly other) contexts. You might also want to consider some of the following questions: To what extent is truth absolute? To what extent can truth change with time? To what extent is truth based on opinion? To what extent are people free to accept as true anything they wish?