Math 3210-3

HW 27

Solutions

Applications of Uniform Convergence

1. Let $f_n(x) = \dfrac{nx}{1+nx}$ for $x \in [0,1]$. Show that the sequence (f_n) does not converge uniformly on $[0,1]$ by using Theorem 118.

   Proof: Notice that for $x \in (0,1]$, we have $f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{nx}{1+nx} = 1$. And if $x = 0$ we have $f(0) = \lim f_n(0) = 0$. So $f(x) = \begin{cases} 1 & 0 < x \leq 1 \\ 0 & x = 0 \end{cases}$.

   We know $f_n(x)$ is continuous for all n and $x \in [0,1]$, but $f(x)$ is not continuous. So by Theorem 118 we know f_n does not uniformly converge to f on $[0,1]$.

   □

2. Let $f_n(x) = \dfrac{n + \sin nx}{3n + \sin^2 nx}$ for $x \in \mathbb{R}$.

   (a) Show that (f_n) converges uniformly on $\mathbb{R}$.

   Proof: For $x \in \mathbb{R}$ we will let $f(x) = \lim_{n\to\infty} f_n(x)$. Notice that for $x \in \mathbb{R}$ we have $\frac{n-1}{3n+1} \leq f_n(x) \leq \frac{n+1}{3n}$. Also $\lim_{n\to\infty} \frac{n-1}{3n+1} = \lim_{n\to\infty} \frac{n+1}{3n} = \frac{1}{3}$, so by the squeeze theorem $f(x) = \lim f_n(x) = \frac{1}{3}$. To prove this convergence is uniform, let $\epsilon > 0$ and let $N = \frac{4}{9\epsilon}$. Then for $x \in \mathbb{R}$ and $n > N$ we have

   $$\begin{aligned} \left|f_n(x) - \frac{1}{3}\right| &= \left|\frac{n + \sin nx}{3n + \sin^2 nx} - \frac{1}{3}\right| \\ &= \left|\frac{3n - 3\sin nx - 3n - \sin^2 nx}{3(3n + \sin^2 nx)}\right| \\ &= \left|\frac{3\sin nx - \sin^2 nx}{3(3n + \sin^2 nx)}\right| \\ &\leq \frac{4}{3(3n + \sin^2 nx)} \\ &= \frac{4}{3(3n + \sin^2 nx)} \\ &\leq \frac{4}{9n} \\ &< \frac{4}{9N} \\ &= \epsilon \end{aligned}$$

   Therefore (f_n) uniformly converges on $\mathbb{R}$.

   □

   (b) Use Theorem 119 to evaluate $\displaystyle\lim_{n\to\infty} \int_0^\pi f_n(x)\,dx$.

   Proof: Since we proved the convergence is uniform, we can apply Theorem 119 to get $\displaystyle\lim_{n\to\infty} \int_0^\pi f_n(x)\,dx = \int_0^\pi \lim_{n\to\infty} f_n(x)\,dx = \int_0^\pi \frac{1}{3}\,dx = \frac{\pi}{3}$.

   □

3. Using Corollary 9, integrate the geometric series

   $$\frac{1}{1-t} = 1 + t + \cdots + t^n + \cdots$$

   term by term from $-x$ to x, where $x \in (-1,1)$, and obtain a series for $\log\left(\dfrac{1+x}{1-x}\right)$.

Proof: Since $x \in (-1,1)$, we see $[-x,x] \subset (-1,1)$, so $|t^n| \leq x^n < 1$. Thus by the Weierstrass M-test $\sum t^n$ converges uniformly on $[-x,x]$. Thus we can apply Corollary 9 to integrate the series term by term. We have

$$\begin{aligned}
\log\left(\frac{1+x}{1-x}\right) &= \int_{-x}^{x} \frac{1}{1-t}\,dt \\
&= \int_{-x}^{x} \sum_{n=0}^{\infty} t^n\,dt \\
&= \sum_{n=0}^{\infty} \int_{-x}^{x} t^n\,dt \\
&= \sum_{n=0}^{\infty} \frac{1}{n+1} t^{n+1}|_{-x}^{x} \\
&= \sum_{n=0}^{\infty} \left[\frac{1}{n+1}(x^{n+1} - (-x)^{n+1})\right] \\
&= \sum_{n \text{ odd}}^{\infty} \frac{1}{n}(2x^n) \\
&= \sum_{n=1}^{\infty} \frac{1}{2n-1}(2x^{2n-1})
\end{aligned}$$

□

Uniform Convergence of Power Series

4. (a) Find the function given by the series $\sum_{n=1}^{\infty} n^2 x^n$ for $|x| < 1$.

Proof: We want to find the function whose series is given by $\sum_{n=1}^{\infty} n^2 x^n = x + 4x^2 + 9x^3 + 16x^4 + \cdots$ for $|x| < 1$. We know the following:

$$\begin{aligned}
\frac{1}{1-x} &= 1 + x + x^2 + x^3 + \cdots && \text{differentiate to get} && (1) \\
\frac{1}{(1-x)^2} &= 1 + 2x + 3x^2 + 4x^3 + \cdots && \text{differentiate again to get} && (2) \\
\frac{2}{(1-x)^3} &= 2 + 6x + 12x^2 + 20x^3 + \cdots && \text{divide by 2 to get} && (3) \\
\frac{1}{(1-x)^3} &= 1 + 3x + 6x^2 + 10x^3 + \cdots && \text{multiply by } x \text{ to get} && (4) \\
\frac{x}{(1-x)^3} &= x + 3x^2 + 6x^3 + 10x^4 + \cdots && \text{multiply eqn(4) by } x^2 \text{ to get} && (5) \\
\frac{x^2}{(1-x)^3} &= x^2 + 3x^3 + 6x^4 + 10x^5 + \cdots && \text{now add eqn(5) and eqn(6) to get} && (6) \\
\frac{x+x^2}{(1-x)^3} &= x + 4x^2 + 9x^3 + 16x^4 + \cdots && && (7) \\
&= \sum_{n=1}^{\infty} n^2 x^n && && (8)
\end{aligned}$$

This series has the same radius of convergence as the geometric series, so this is for all $|x| < 1$.

□

(b) Evaluate $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ and $\sum_{n=2}^{\infty} \frac{n^2}{2^n}$.

Proof: Since $f(x) = \frac{x+x^2}{(1-x)^3} = \sum_{n=1}^{\infty} n^2 x^n$, then $\sum_{n=1}^{\infty} \frac{n^2}{2^n} = f(\frac{1}{2}) = 6$ and $\sum_{n=2}^{\infty} \frac{n^2}{2^n} = f(\frac{1}{2}) - \frac{1}{2} = \frac{11}{2}$.

□

5. (a) Show that $\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$ for $|x| < 1$.

Proof: For $|x| < 1$ we have the following:

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots \qquad \text{replace } x \text{ with } -x \tag{9}$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots \qquad \text{replace } x \text{ with } x^2 \tag{10}$$

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots \tag{11}$$

$$= \sum_{n=0}^{\infty} (-1)^n x^{2n} \tag{12}$$

□

(b) Show that $\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$ for $|x| < 1$.

Proof: Because power series converge uniformly, we can apply Theorem 119 to get

$$\arctan x = \int_0^x \frac{1}{1+t^2}\, dt \tag{13}$$

$$= \int_0^x (1 - t^2 + t^4 - t^6 + \cdots)\, dt \tag{14}$$

$$= x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots \tag{15}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \tag{16}$$

□

(c) Show that the series for $\arctan x$ in part(b) also holds when $x = 1$.

Proof: When $x = 1$ we have $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1}$ which converges by the alternating series test.

□

(d) Use part (c) to find a series whose sum is π.

Proof: Since $\arctan 1 = \frac{\pi}{4}$ we see $\pi = 4 \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 4\left(1 - \frac{1}{5} + \cdots\right)$.

□

6. (a) Show that $\int_0^x \arctan t\, dt = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{2n}}{2n(2n-1)}$ for $|x| < 1$.

Proof: From above we see $\arctan x = x - \dfrac{x^3}{3} + \cdots = \displaystyle\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}$ for $|x| < 1$, so we can integrate term by term to get:

$$\begin{aligned}
\int_0^x \arctan t \, dt &= \int_0^x \left[\sum_{n=0}^{\infty}(-1)^n \frac{t^{2n+1}}{2n+1}\right] dt \\
&= \sum_{n=0}^{\infty}(-1)^n \left[\int_0^x \frac{t^{2n+1}}{2n+1}\, dt\right] \\
&= \sum_{n=0}^{\infty}(-1)^n \left[\frac{t^{2n+2}}{(2n+1)(2n+2)}\bigg|_0^x\right] \\
&= \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+2}}{(2n+1)(2n+2)} \\
&= \sum_{n=1}^{\infty}(-1)^{n-1} \frac{x^{2n}}{(2n)(2n-1)}
\end{aligned}$$

□

(b) Show that the formula in part (a) also holds for $x = 1$.

Proof: At $x = 1$ the above formula gives us $\displaystyle\int_0^x \arctan t \, dt = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n(2n-1)} = \frac{1}{2} - \frac{1}{12} + \frac{1}{30} - \frac{1}{56} + \cdots$ which converges by the alternating series test. Thus the formula holds for $x = 1$.

□

(c) Assuming that the series $1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} - \frac{1}{7} + \cdots$ is convergent, use part (b) to find its value.

Proof: Notice that $1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} - \frac{1}{7} + \cdots = (1 - \frac{1}{2}) + (-\frac{1}{3} + \frac{1}{4}) + (\frac{1}{5} - \frac{1}{6}) + (-\frac{1}{7} + \cdots = \frac{1}{2} - \frac{1}{12} + \frac{1}{30} - \frac{1}{56} + \cdots = \int_0^1 \arctan t \, dt = t \arctan t - \frac{1}{2}\ln(1 + t^2)\Big|_0^1 = \frac{\pi}{4} - \frac{1}{2}\ln 2$

□