Math 3210-3
HW 26
Solutions

Power Series

1. Find the radius of convergence R and the interval of convergence C for each series:

   (a) $\sum \frac{n^2}{2^n} x^n$

   $$R = \lim \frac{n^2}{2^n} \frac{2^{n+1}}{(n+1)^2} = \lim \frac{2n^2}{(n+1)^2} = 2$$

   At $x = -2$, our series is $\sum \frac{n^2}{2^n}(-2)^n = \sum (-1)^n n^2$ which diverges. At $x = 2$ our series is $\sum \frac{n^2}{2^n}(2^n) = \sum n^2$ which diverges. Thus $C = (-2, 2)$.

   (b) $\sum \frac{(-4)^{-n}}{n} x^n$

   $$R = \lim \frac{4^{n+1}(n+1)}{4^n n} = \lim \frac{4(n+1)}{n} = 4$$

   At $x = -4$ our series is $\sum \frac{(-4)^n}{n}(-4)^n = \sum \frac{1}{n}$ which diverges. At $x = 4$ our series is $\sum \frac{(-1)^n}{n}$ which converges, so $C = (-4, 4]$.

   (c) $\sum (2^{-n})(x-5)^{2n}$

   First, let $y = (x-5)^2$. Then we will consider the series $\sum (2^{-n}) y^n$. For this series we have

   $$R = \lim \frac{2^{n+1}}{2^n} = 2$$

   Thus the radius of convergence for our original series is $R = 2$. Thus the series converges for $|y| = |(x-5)^2| = (x-5)^2 < 2$. This means it converges for $5 - \sqrt{2} < x < 5 + \sqrt{2}$. Now to check the endpoints. At $x = 5 - \sqrt{2}$, our series is $\sum \frac{1}{2^n}(-\sqrt{2})^{2n} = \sum (-1)^n$ which diverges. At $x = 5 + \sqrt{2}$ our series is $\sum \frac{1}{2^n}(\sqrt{2})^{2n} = \sum 1$ which also diverges. Thus $C = (5 - \sqrt{2}, 5 + \sqrt{2})$.

2. Find the radius of convergence for $\sum \frac{(3n)!}{(n!)^2} x^n$.

   We will use the ratio test. $R = \lim \frac{(3n)!}{(n!)^2} \cdot \frac{[(n+1)!]^2}{[3(n+1)]!} = \lim \frac{(n+1)^2}{(3n+1)(3n+2)(3n+3)} = 0.$

3. Suppose that the series $\sum a_n x^n$ has radius of convergence 2. Find the radius of convergence of each series, where k is a fixed positive integer.

   (a) $\sum a_n^k x^n$

   Notice that if $R = 2$ is the radius of convergence of $\sum a_n x^n$, then $\lim \frac{|a_n|}{|a_{n+1}|} = 2$. Thus the radius of convergence of $\sum a_n^k x^n$ is $R' = \lim \frac{|a_n^k|}{|a_{n+1}^k|} = \lim \left(\frac{|a_n|}{|a_{n+1}|}\right)^k = \left(\lim \frac{|a_n|}{|a_{n+1}|}\right)^k = 2^k$.

   (b) $\sum a_n x^{kn}$

   Let $y = x^k$. Then our new series is $\sum a_n y^n$ which has radius of converence 2, so $\sum a_n x^{kn}$ converges for $|y| = |x^k| < 2$. Thus the radius of convergence of our original series is $R'' = 2^{1/k}$.

   (c) $\sum a_n x^{n^2}$

   Let $y = x^n$. Then our new series is $\sum a_n y^n$ which has radius of convergence 2, so our original series will converge if $|x^n| < 2$ for all n. So it will converge if $|x| < 2^{1/n}$ for all n. But $\lim_{n\to\infty} 2^{1/n} = 1$, so our series will converge for $|x| < 1$. Thus our radius of convergence is 1.

4. Prove that the series $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} n a_n x^n$ have the same radius of convergence (finite or infinite).

Proof: Let $\alpha = \lim |a_n|^{\frac{1}{n}}$. Then $R = \begin{cases} \frac{1}{\alpha} & \text{if } 0 < \alpha < +\infty \\ 0 & \text{if } \alpha = +\infty \\ \infty & \text{if } \alpha = 0 \end{cases}$ as in Theorem 113. Also notice that $\lim |n a_n|^{\frac{1}{n}} = \lim n^{\frac{1}{n}} |a_n|^{\frac{1}{n}} = \alpha$ since $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$ as we proved in class. Thus the two series have the same radius of convergence.

□

Pointwise and Uniform Convergence

5. Let $f_n(x) = \frac{x^n}{n}$ for $x \in [-1,1]$. Find $f(x) = \lim f_n(x)$ and determine whether or not the convergence is uniform on $[-1,1]$. Justify your answer.

Proof: If $x \in [-1,1]$, then $\frac{-1}{n} \le \frac{x^n}{n} \le \frac{1}{n}$, and $\lim_{n\to\infty} \frac{-1}{n} = \lim_{n\to\infty} \frac{1}{n} = 0$, so by the squeeze theorem, $f(x) = \lim f_n(x) = 0$ for $x \in [-1,1]$.

Let $\epsilon > 0$ and let $N = \frac{1}{\epsilon}$. Then for $n > N$ and $x \in [-1,1]$ we have $|f_n(x) - f(x)| = |\frac{x^n}{n}| \le \frac{1}{n} < \frac{1}{N} = \epsilon$. Thus (f_n) converges uniformly to f on $[-1,1]$.

□

6. Let $f_n(x) = \frac{x}{x+n}$ for $x \ge 0$.

   (a) Show that $f(x) = \lim f_n(x) = 0$ for all $x \ge 0$.
   Proof: For a fixed $x \ge 0$ we have $f(x) = \lim_{n\to\infty} f_n(x) = 0$.

   □

   (b) Show that if $t > 0$, the convergence is uniform on $[0,t]$.

   Proof: If $x \in [0,t]$ for some fixed $t > 0$, then given $\epsilon > 0$ let $N = \max\{\frac{t - \epsilon t}{\epsilon}, 1\}$. Then if $n > N$ and $x \in [0,t]$, we have $|f_n(x) - f(x)| = |\frac{x}{x+n}| \le \frac{t}{t+n} \le \frac{t}{t+N} = \epsilon$. Therefore (f_n) converges uniformly to f on $[0,t]$.

   □

   (c) Show that the convergence is not uniform on $[0,\infty)$.
   Proof: Let $\epsilon = \frac{1}{2}$. Then given $x > n$ we have $|f_n(x) - f(x)| = |\frac{x}{x+n}| > \frac{x}{2x} = \frac{1}{2}$. Thus (f_n) does not converge uniformly on $[0,\infty)$.

   □

7. If (f_n) and (g_n) converge uniformly on a set S, prove that $(f_n + g_n)$ converges uniformly on S.

Proof: Let $\epsilon > 0$. Then there is some N_1 such that for $n, m > N_1$ and $x \in S$ we have $|f_n(x) - f_m(x)| < \frac{\epsilon}{2}$. We also have some N_2 such that for $n, m > N_2$ and $x \in S$, then $|g_n(x) - g_m(x)| < \frac{\epsilon}{2}$. Let $N = \max\{N_1, N_2\}$. Then for $n, m > N$ and $x \in S$ we have $|(f_n(x) + g_n(x)) - (f_m(x) + g_m(x))| = |(f_n(x) - f_m(x)) + (g_n(x) - g_m(x))| \le |f_n(x) - f_m(x)| + |g_n(x) - g_m(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. Therefore $(f_n + g_n)$ converges uniformly on S.

□

8. Determine whether or not the given series of functions converges uniformly on the indicated set. Justify your answers.

(a) $\sum n^{-x}$ for $x > \sqrt{2}$

Proof: Since $x > \sqrt{2}$ we have $|f_n(x)| = |\frac{1}{n^x}| < \frac{1}{n^{\sqrt{2}}}$. Also $\sum \frac{1}{n^{\sqrt{2}}}$ converges since $\sqrt{2} > 1$, so by the Weierstrass M-test, $\sum n^{-x}$ converges uniformly for $x > \sqrt{2}$.

□

(b) $\sum \dfrac{x^2}{n^2}$ for $x \geq 5$

Proof: Notice that the nth partial sum $s_n = \displaystyle\sum_{i=1}^{n} \frac{x^2}{i^2} = x^2(1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2})$. So given $\epsilon = 1$ and any $n, m \in \mathbb{N}$ with $n > m \geq 1$ let $x > n$. Then $|s_n - s_m| = |x^2(\frac{1}{m^2} + \frac{1}{(m+1)^2} + \cdots + \frac{1}{n^2}| > n^2(\frac{1}{m^2} + \cdots + \frac{1}{n^2}) > 1 = \epsilon$. Therefore $\sum f_n$ is not uniformly convergent on the set $x \geq 5$.

□