Math 3210-3
HW 25
Solutions

Convergence Tests

1. Determine the values of p for which the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$ converges.

   Proof: We will use the integral test. Let $f(x) = \dfrac{1}{x(\log x)^p}$. So we want to compute $\displaystyle\int_2^{\infty} f(x)\,dx$. Let's make the substitution $u = \log x$ so $du = \dfrac{1}{x}\,dx$ so our integral becomes $\displaystyle\int \frac{1}{xu^p}\cdot x\,du = \int u^{-p}\,du$. There are two cases to consider:
   Case 1: If $p = 1$ then $\displaystyle\int f(x)\,dx = \log u = \log(\log x)|_2^{\infty} = \lim_{b\to\infty} \log(\log b) - \log(\log 2) = \infty$. Thus the series diverges when $p = 1$.
   Case 2: If $p \neq 1$, then $\displaystyle\int f(x)\,dx = \frac{1}{1-p}u^{-p+1} = \frac{1}{1-p}(\log x)|_2^{\infty} = \lim_{b\to\infty} \frac{1}{1-p}[(\log b)^{-p+1} - (\log 2)^{-p+1}]$. We know this converges if $-p+1 < 0$ and diverges for $-p+1 > 0$.
   Therefore the series converges if $p > 1$ and diverges otherwise.

   □

2. Determine whether each series converges conditionally, converges absolutely, or diverges. Justify your answers.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\log n}$

   Proof: For this series, $a_n = \dfrac{(-1)^n}{\log n}$. So $|a_n| = \dfrac{1}{\log n}$. Notice that $\log n < n$, so $\frac{1}{\log n} > \frac{1}{n}$. Thus by the comparison test, $\sum |a_n|$ diverges. On the other hand $\lim |a_n| = 0$, so $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\log n}$ converges conditionally.

   □

   (b) $\displaystyle\sum \frac{(-2)^n}{n^2}$

   Proof: Notice $a_n = \dfrac{(-2)^n}{n^2}$ and $\lim |a_n| = \lim \dfrac{2^n}{n^2} = \lim \dfrac{2^n \log 2}{2n} = \lim \dfrac{2^n(\log 2)^2}{2} = \infty$. Thus the series diverges.

   □

   (c) $\displaystyle\sum \frac{(-3)^n}{n!}$

   Proof: For this series $a_n = \dfrac{(-3)^n}{n!}$. Using the ratio test we see $\lim \dfrac{|a_{n+1}|}{|a_n|} = \lim \dfrac{3^{n+1}}{(n+1)!}\cdot\dfrac{n!}{3^n} = \lim \dfrac{3}{n+1} = 0$. Thus this series converges absolutely.

   □

   (d) $\displaystyle\sum \left(\frac{1}{\sqrt{n}} - \frac{1}{n}\right)$

   Proof: Notice that $\dfrac{1}{\sqrt{n}} - \dfrac{1}{n} = \dfrac{\sqrt{n}-1}{n} > \dfrac{1}{n}$ for $n > 4$. Thus by the comparison test this series diverges.

□

3. Find an example to show that the convergence of $\sum a_n$ and the convergence of $\sum b_n$ do not necessarily imply the convergence of $\sum(a_n b_n)$.

   Let $a_n = \dfrac{(-1)^n}{n}$ and $b_n = \dfrac{(-1)^n}{\log n}$. By the alternating series tests $\sum a_n$ and $\sum b_n$ converge, but $\sum a_n b_n = \sum \dfrac{1}{n \log n}$. We showed in problem 4 that this series diverges.

4. Show that the series
$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{2^2} + \frac{1}{5} - \frac{1}{2^3} + \frac{1}{7} - \frac{1}{2^4} + \cdots$$
   diverges. Why doesn't this contradict the alternating series test?

   Proof: Let $\sum a_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{2^2} + \frac{1}{5} - \frac{1}{2^3} + \frac{1}{7} - \frac{1}{2^4} + \cdots$ and let (s_n) be the sequence of its partial sums. We will show that (s_n) is not a bounded sequence, so $\sum a_n$ diverges. Notice that $a_n = \begin{cases} \frac{1}{2^{n/2}} & \text{for } n \text{ even} \\ \frac{1}{n} & \text{for } n \text{ odd} \end{cases}$. So we can write

$$s_n = \sum_{i=1}^{n} a_n = \sum_{i=1}^{\frac{n-1}{2}} \frac{1}{2i+1} - \sum_{i=1}^{\frac{n-1}{2}} \frac{1}{2^i} = b_m - c_m$$

   where (b_n) and (c_n) are the sequences of partial sums of the odd terms and even terms of a_n, respectively, and $m = \frac{n-1}{2}$.

   Notice that $\dfrac{1}{2i+1} < \dfrac{1}{2i}$ and $\sum \frac{1}{2i} = \frac{1}{2} \sum \frac{1}{i}$ which diverges, so by the comparison test (b_n) diverges, so $\sum \frac{1}{2i+1}$ diverges. In other words, (b_n) is an unbounded sequence.

   Also $\sum \frac{1}{2^i}$ is a geometric series, so it converges and $\sum \frac{1}{2^i} = \frac{1}{1-\frac{1}{2}} = 2$. Since (c_n) is the sequence of the partial sums of this series, (c_n) is an increasing sequence, so $c_n < 2$ for all n.

   Finally, let $M \in \mathbb{R}$. There exists some N such that for all $n > N$, we have $b_n > M + 2$. This implies that $s_n = \sum_{i=1}^{n} a_i = b_n - c_n > M + 2 - 2 = M$. Therefore the sequence of partial sums (s_n) is unbounded, so $\sum a_n$ diverges.

□

   This does not contradict the alternating series test since $(|a_n|)$ is not a decreasing sequence.