Math 3210-3

HW 24

Solutions

Convergence of Infinite Series

1. Find the sum of each series:

   (a) $\displaystyle\sum_{n=3}^{\infty}\left(\frac{1}{2}\right)^n$

   $$\sum_{n=3}^{\infty}\left(\frac{1}{2}\right)^n = \sum_{n=0}^{\infty}\frac{1}{8}\left(\frac{1}{2}\right)^n = \frac{1}{8}\sum\left(\frac{1}{2}\right)^n = \frac{1}{8}\frac{1}{1-\frac{1}{2}} = \frac{1}{4}$$

   (b) $\displaystyle\sum_{n=1}^{\infty}\left(-\frac{2}{3}\right)^n$

   $$\sum_{n=1}^{\infty}\left(-\frac{2}{3}\right)^n = \sum_{n=0}^{\infty}\left(-\frac{2}{3}\right)^n - 1 = \frac{1}{1-\frac{-2}{3}} - 1 = \frac{-2}{5}$$

   (c) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n(n+1)(n+2)}$

   Notice that $\displaystyle\frac{1}{n(n+1)(n+2)} = \frac{1}{2}\left(\frac{2}{n(n+1)(n+2)}\right) = \frac{1}{2}\left(\frac{1}{n(n+1)} - \frac{1}{(n+1)(n+2)}\right)$. So we have:

   $$\begin{aligned} s_n &= \sum_{i=1}^{n}\frac{1}{i(i+1)(i+2)} \\ &= \sum\frac{1}{2}\left(\frac{1}{i(i+1)} - \frac{1}{(i+1)(i+2)}\right) \\ &= \frac{1}{2}\left[\left(\frac{1}{1\cdot 2} - \frac{1}{2\cdot 3}\right) + \left(\frac{1}{2\cdot 3} - \frac{1}{3\cdot 4}\right) + \cdots + \left(\frac{1}{n(n+1)} - \frac{1}{(n+1)(n+2)}\right)\right] \\ &= \frac{1}{2}\left(\frac{1}{2} - \frac{1}{(n+1)(n+2)}\right) \end{aligned}$$

   So $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n(n+1)(n+2)} = \lim_{n\to\infty} s_n = \frac{1}{4}$.

2. Determine whether or not the series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt{n+1}+\sqrt{n}}$ converges. Justify your answer.

   Notice that $\displaystyle\frac{1}{\sqrt{n+1}+\sqrt{n}} = \frac{1}{\sqrt{n+1}+\sqrt{n}}\cdot\frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n+1}-\sqrt{n}} = \frac{\sqrt{n+1}-\sqrt{n}}{1} = \sqrt{n+1}-\sqrt{n}$. We have

   $$\begin{aligned} s_n &= \sum_{i=1}^{n}\sqrt{i+1}-\sqrt{i} \\ &= (\sqrt{2}-\sqrt{1}) + (\sqrt{3}-\sqrt{2}) + (\sqrt{4}-\sqrt{3}) + \cdots + (\sqrt{n}-\sqrt{n-1}) + (\sqrt{n+1}-\sqrt{n}) \\ &= -1 + \sqrt{n+1} \end{aligned}$$

   Thus $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt{n+1}+\sqrt{n}} = \lim_{n\to\infty} s_n = \infty$, so the series diverges.

3. Prove that if $\sum|a_n|$ converges and (b_n) is a bounded sequence, then $\sum a_n b_n$ converges.

   Proof: Since (b_n) is bounded, there is some $B\in\mathbb{R}$ such that $|b_n| < B$ for all n. Let $\epsilon > 0$. Then there is some N such that if $n \geq m > N$, then $||a_m| + |a_{m+1}| + \cdots + |a_n|| < \frac{\epsilon}{B}$. Thus $|a_m b_m + a_{m+1}b_{m+1} +$

$\cdots + a_n b_n| \le |a_m|B + |a_{m+1}|B + \cdots + |a_n|B < B[|a_m| + \cdots + |a_n|] \le B||a_m| + \cdots + |a_n|| < B\frac{\epsilon}{B} = \epsilon$. Therefore by Theorem 107, $\sum a_n b_n$ converges.

$\square$