Math 3210-3
HW 23
Solutions

The Fundamental Theorem of Calculus

1. ♣ Calculate

   (a) $\lim_{x\to 0}\left(\frac{1}{x}\right)\int_0^x e^{t^2}\,dt$

   Proof: Since this limit has the indeterminant form $\frac{0}{0}$, we can apply L'Hospital's Theorem to get:

   $$\lim_{x\to 0}\frac{\int_0^x e^{t^2}\,dt}{x} = \lim_{x\to 0} e^{x^2} = 1$$

   □

   (b) $\lim_{h\to 0}\left(\frac{1}{h}\right)\int_3^{3+h} e^{t^2}\,dt$

   Proof: We will apply L'Hospital's Theorem, but first, let's change the index value. Let $x = 3+h$. Then we have

   $$\lim_{h\to 0}\frac{\int_3^{3+h} e^{t^2}\,dt}{h} = \lim_{x\to 3}\frac{\int_0^x e^{t^2} - \int_0^3 e^{t^2}\,dt}{x-3} = \lim_{x\to 3}\frac{e^{x^2}-0}{1} = e^{3^2} = e^9$$

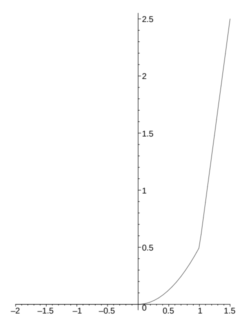
   □

2. ♣ Let f be defined as follows: $f(t) = 0$ for $t < 0$; $f(t) = t$ for $0 \le t \le 1$; $f(t) = 4$ for $t > 1$.

   (a) Determine the function $F(x) = \int_0^x f(t)\,dt$.

   $$\begin{aligned} F(x) &= \begin{cases} 0 & \text{if } x \le 0 \\ \int_0^x t\,dt & \text{if } 0 < x \le 1 \\ \int_0^1 t\,dt + 4(x-1) & \text{if } x > 1 \end{cases} \\ &= \begin{cases} 0 & \text{if } x \le 0 \\ \frac{x^2}{2} & \text{if } 0 < x \le 1 \\ 4x - \frac{7}{2} & \text{if } x > 1 \end{cases} \end{aligned}$$

   (b) Sketch F. Where is F continuous?

   F is continuous on all of $\mathbb{R}$.

   (c) Where is F differentiable? Calculate F' at the points of differentiability.

   The only points we need to worry about are $x = 0, 1$. Let's compute the derivative of F at $x = 0$. We have:

   $$F'(0) = \lim_{x\to 0}\frac{F(x)-0}{x} = 0$$

So $F'(0) = 0$. At $x = 1$ we need to compute a two-sided limit:

$$\lim_{x\to 1^+} \frac{F(x) - \frac{1}{2}}{x-1} = \lim_{x\to 1^+} \frac{4(x-1)}{x-1} = 4$$

$$\lim_{x\to 1^-} \frac{F(x) - \frac{1}{2}}{x-1} = \lim_{x\to 1^-} \frac{\frac{x^2}{2} - \frac{1}{2}}{x-1} = \lim_{x\to 1^-} \frac{\frac{1}{2}(x-1)(x+1)}{x-1} = 1$$

Thus F is not differentiable at $x = 1$. For $x \neq 1$, we have

$$F'(x) = \begin{cases} 0 & x \leq 0 \\ x & 0 < x < 1 \\ 4 & x > 1 \end{cases}$$

3. ♣ Let $f : [0,1] \to \mathbb{R}$ be a continuous function with continuous second derivative f'', and $f(0) = f'(1) = 0$. Prove that if $\displaystyle\int_0^1 f(x)f''(x)\,dx = 0$, then $f \equiv 0$. (Hint: Integration by parts.)

Proof: We will perform integration by parts on the above integral. With the the above assumptions, we have:

$$0 = \int_0^1 ff'' = ff'(1) - ff'(0) - \int_0^1 f'f' = -\int_0^1 (f')^2$$

Thus $\int_0^1 (f')^2 = 0$. If we could show $f'(x) = 0$ for all $x \in [0,1]$ then by Theorem 86, f will be constant on $[0,1]$ which implies $f \equiv 0$ since $f(0) = 0$. Suppose there is some $x \in [0,1]$ such that $|f'(x)| > 0$. Then since f' is differentiable, f' is continuous, so there is a neighborhood $U = (u_1, u_2) \subseteq [0,1]$ such that $x \in U$ and $f'(x) > 0$ or $f'(x) < 0$, but not both, for all $x \in U$. This means that $[f'(x)]^2 > 0$ for all $x \in U$. Let $P = \{u_1, u_2\}$, so P is a partition of U, and $m(f,P) > 0 \implies \int_{u_1}^{u_2} (f')^2 > m(f,P)(u_2 - u_1) > 0$. Also, since $(f')^2 \geq 0$ for all $x \in [0,1]$, we know $\int_0^1 (f')^2 \geq 0$. Thus $0 = \int_0^1 (f')^2 = \int_0^{u_1} (f')^2 + \int_{u_1}^{u_2} (f')^2 + \int_{u_2}^1 (f')^2 > 0$, a contradition. So $f'(x) = 0$ for all $x \in [0,1]$. Therefore $f \equiv 0$.

□