Math 3210-3
HW 22
Solutions

Properties of the Riemann Integral

1. ♣ A function f on $[a,b]$ is called a *step function* if there exists a partition $P = \{a = u_0 < u_1 < \cdots < u_m = b\}$ of $[a,b]$ such that f is constant on each interval (u_{j-1}, u_j), say $f(x) = c_j$ for $x \in (u_{j-1}, u_j)$.

   (a) Show that a step function f is integrable and evaluate $\int_a^b f$.
   Proof: Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a,b]$ such that f is constant on (x_{i-1}, x_i) for each $i = 1, \ldots, n$. Then we can say $f(x) = c_i$ for $x \in (x_{i-1}, x_i)$.
   I claim that f is integrable on $[x_{i-1}, x_i]$ for each $i = 1, \ldots, n$. To prove this claim, notice that if $g(x) = c_i$ for all $x \in [x_{i-1}, x_i]$, then $f(x) = g(x)$ except possibly at two points. Also g is monotonic, so by Theorem 98 g is integrable on $[x_{i-1}, x_i]$. By problem 6 on homework 21, we can conclude that f is integrable on $[x_{i-1}, x_i]$ for each $i = 1, \ldots, n$. Thus by Theorem 100 f is integrable on $[a,b]$ and $\displaystyle\int_a^b f = \sum_{i=1}^n \int_{x_{i-1}}^{x_i} f = \sum_{i=1}^n c_i \Delta x_i$.

   □

   (b) Given $P(x) = \begin{cases} 15 & \text{if } 0 \le x < 1 \\ 15 + 13n & \text{if } n \le x < n+1 \end{cases}$, evaluate $\int_0^4 P(x)\,dx$. Note: $P(x)$ is called the postage-stamp function. Do you see why?
   Proof: By part (a), we see $\displaystyle\int_0^4 P(x)\,dx = \int_0^1 15\,dx + \int_1^2 (15+13)\,dx + \int_2^3 (15+13(2))\,dx + \int_3^4 (15+13(3))\,dx = 15 + 28 + 41 + 54 = 138.$

   □

2. ♣ Prove that if f is integrable on $[a,b]$ then so is f^2. (Hint: If $|f(x)| \le M$ for all $x \in [a,b]$, then show $|f^2(x) - f^2(y)| \le 2M|f(x) - f(y)|$ for all $x, y \in [a,b]$. Then use this to estimate $U(f^2,P) - L(f^2,P)$ in terms of $U(f,P) - L(f,P)$ for a given partition P.)

   Proof: Let $\epsilon > 0$. Since f it is bounded. Let $M \in \mathbb{R}$ such that $|f(x)| < M$ for all $x \in [a,b]$. From Theorem 97, there is a partition P of $[a,b]$ such that $U(f,P) - L(f,P) < \frac{\epsilon}{2M}$. I claim that with this partition we have $U(f^2,P) - L(f^2,P) < \epsilon$, so by Theorem 97, f^2 will be integrable. To this end, notice that $|f^2(x) - f^2(y)| = |f(x) + f(y)||f(x) - f(y)| \le (|f(x)| + |f(y)|)|f(x) - f(y)| < 2M|f(x) - f(y)|$ for all $x, y \in [a,b]$.

   Thus we have

$$\begin{aligned} U(f^2,P) - L(f^2,P) &= \sum |f^2(x_i) - f^2(y_i)||x_i - y_i| \\ &\le \sum 2M|f(x_i) - f(y_i)||x_i - y_i| \\ &= 2M(U(f,P) - L(f,P)) \\ &< 2M\frac{\epsilon}{2M} \\ &= \epsilon \end{aligned}$$

   Therefore f^2 is integrable.

   □

3. Prove that if f and g are integrable on $[a,b]$, then so is fg. (Hint: Use the previous problem to write fg as the sum of two functions which you know are integrable.)

   Proof: Notice that f and g are integrable, so $f+g$, f^2, g^2, and $(f+g)^2$ are integrable by the previous problem and Theorem 100. Thus $\frac{1}{2}[(f+g)^2 - f^2 - g^2] = fg$ is integrable.

$\square$

4. ♣ Find an example of a function $f : [0, 1] \to \mathbb{R}$ such that f is not integrable on $[0, 1]$ by $|f|$ is integrable on $[0, 1]$.

Proof: Let $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ -1 & \text{if } x \notin \mathbb{Q} \end{cases}$. Let $P = \{x_0, \ldots, x_n\}$ be any partition of $[0, 1]$. Then $M_i = 1$ and $m_i = -1$ for all $i = 1, \ldots n$. Thus $U(f, P) = 1$ and $L(f, P) = -1$ for all P. Thus $U(f) = 1$ and $L(f) = -1$. Thus f is not integrable.

On the other hand, $|f|(x) = 1$ for all $x \in [0, 1]$. Since $|f|$ is a continuous function, $|f|$ is integrable on $[0, 1]$ by Theorem 98.

$\square$

5. ♣ Suppose that f and g are continuous function on $[a, b]$ such that $\int_a^b f = \int_a^b g$. Prove that there exists $x \in [a, b]$ such that $f(x) = g(x)$.

Proof: Let $h(x) = f(x) - g(x)$. Then h is continuous by Theorem 71, and h assumes its max and min values on $[a, b]$ by Corollary 5. In other words, there is some $x_1, x_2 \in [a, b]$ such that $h(x_1) = m \leq < h(x) \leq M = h(x_2)$ for all $x \in [a, b]$. If P is the partition $P = \{a, b\}$, then $L(h, P) = m(b - a)$ and $U(h, P) = M(b - a)$, so we have $L(h, P) = m(b - a) \leq \displaystyle\int_a^b h \leq M(b - a) = U(f, P) \Longrightarrow h(x_1) = m \leq \dfrac{1}{b - a} \displaystyle\int_a^b h < M = h(x_2)$. Thus we can apply the intermediate value theorem to get some $x \in [a, b]$ such that $h(x) = \frac{1}{b-a} \int_a^b h = \frac{1}{b-a} \int_a^b (f - g) = 0$. Thus $f(x) = g(x)$ for some $x \in [a, b]$.

$\square$