Math 3210-3
HW 22
Due Friday, November 16, 2007

Properties of the Riemann Integral

1. ♣ A function f on $[a,b]$ is called a *step function* if there exists a partition $P = \{a = u_0 < u_1 < \cdots < u_m = b\}$ of $[a,b]$ such that f is constant on each interval (u_{j-1}, u_j), say $f(x) = c_j$ for $x \in (u_{j-1}, u_j)$.
   (a) Show that a step function f is integrable and evaluate $\int_a^b f$.
   (b) Given $P(x) = \begin{cases} 15 & \text{if } 0 \le x < 1 \\ 15 + 13n & \text{if } n \le x < n+1 \end{cases}$, evaluate $\int_0^4 P(x)\,dx$. Note: $P(x)$ is called the postage-stamp function. Do you see why?
2. ♣ Prove that if f is integrable on $[a,b]$ then so is f^2. (Hint: If $|f(x)| \le M$ for all $x \in [a,b]$, then show $|f^2(x) - f^2(y)| \le 2M|f(x) - f(y)|$ for all $x, y \in [a,b]$. Then use this to estimate $U(f^2, P) - L(f^2, P)$ in terms of $U(f,P) - L(f,P)$ for a given partition P.)
3. Prove that if f and g are integrable on $[a,b]$, then so is fg. (Hint: Use the previous problem to write fg as the sum of two functions which you know are integrable.)
4. ♣ Find an example of a function $f : [0,1] \to \mathbb{R}$ such that f is not integrable on $[0,1]$ but $|f|$ is integrable on $[0,1]$.
5. ♣ Suppose that f and g are continuous function on $[a,b]$ such that $\int_a^b f = \int_a^b g$. Prove that there exists $x \in [a,b]$ such that $f(x) = g(x)$.