Math 3210-3
HW 20
Solutions

L'Hospital's Rule

1. Evaluate the following limits.

   (a) $\lim\limits_{x\to 0} \dfrac{\tan x - x}{x^3}$

   $\lim\limits_{x\to 0} \dfrac{\tan x - x}{x^3} = \lim\limits_{x\to 0} \dfrac{\sec^2 x - 1}{3x^2} = \lim\limits_{x\to 0} \dfrac{2\sec^2 x \tan x}{6x} = \lim\limits_{x\to 0} \dfrac{4\sec^2 x \tan^2 x + 2\sec^4 x)}{6} = \dfrac{1}{3}$

   (b) $\lim\limits_{x\to 0} \dfrac{\sin x - x}{e^x - 1}$

   $\lim\limits_{x\to 0} \dfrac{\sin x - x}{e^x - 1} = \lim\limits_{x\to 0} \dfrac{\cos x - 1}{e^x} = 0$

   (c) $\lim\limits_{x\to \infty} \dfrac{x^2}{e^x}$

   $\lim\limits_{x\to \infty} \dfrac{x^2}{e^x} = \lim\limits_{x\to \infty} \dfrac{2x}{e^x} = \lim\limits_{x\to \infty} \dfrac{2}{e^x} = 0$

   (d) $\lim\limits_{x\to 0^+} \dfrac{\log \sin x}{\log x}$

   $\lim\limits_{x\to 0^+} \dfrac{\log \sin x}{\log x} = \lim\limits_{x\to 0^+} \dfrac{\frac{1}{\sin x}\cos x}{\frac{1}{x}} = \lim\limits_{x\to 0^+} \dfrac{x\cos x}{\sin x} = \lim\limits_{x\to 0^+} \dfrac{\cos x - x\sin x}{\cos x} = 1$

   (e) $\lim\limits_{x\to \infty} \dfrac{(\log x)^2}{x}$

   $\lim\limits_{x\to \infty} \dfrac{(\log x)^2}{x} = \lim \dfrac{2\log x\left(\frac{1}{x}\right)}{1} = \lim \dfrac{2\log x}{x} = \lim \dfrac{2}{x} = 0$

   (f) $\lim\limits_{x\to 0^+} x^{2x}$

   Let $y = x^{2x}$. Then $\log y = 2x\log x$, so let's compute $\lim\limits_{x\to 0^+} 2x\log x$. We have $\lim 2x\log x = \lim \frac{\log x}{\frac{1}{2x}} = \lim \frac{\frac{1}{x}}{-(2x)^{-2}(2)} = \lim \frac{\frac{1}{x}}{\frac{-2}{4x^2}} = \lim \frac{4x^2}{-2x} = \lim \frac{8x}{-2} = 0$. So $\lim\limits_{x\to 0^+} x^{2x} = \lim e^{2x\log x} = e^{\lim 2x\log x} = e^0 = 1$.

   (g) $\lim\limits_{x\to 0} x\ln x$

   $\lim\limits_{x\to 0} x\ln x = \lim\limits_{x\to 0} \dfrac{\ln x}{\frac{1}{x}}$ and now we can use L'Hospital's rule. We have $\lim\limits_{x\to 0} x\ln x = \lim\limits_{x\to 0} \dfrac{\ln x}{\frac{1}{x}} = \lim\limits_{x\to 0} \dfrac{\frac{1}{x}}{\frac{-1}{x^2}} = \lim\limits_{x\to 0} \dfrac{-x^2}{x} = \lim\limits_{x\to 0} -x = 0$.

   (h) $\lim\limits_{x\to \infty} (\sqrt{x+1} - \sqrt{x})$

   $\lim\limits_{x\to \infty} (\sqrt{x+1} - \sqrt{x}) = \lim\limits_{x\to \infty} \dfrac{(\sqrt{x+1} - \sqrt{x})(\sqrt{x+1} + \sqrt{x})}{\sqrt{x+1} + \sqrt{x}} = \lim\limits_{x\to \infty} \dfrac{x+1-x}{\sqrt{x+1} + \sqrt{x}} = \lim\limits_{x\to \infty} \dfrac{1}{\sqrt{x+1} + \sqrt{x}} = 0$.

2. Indicate what is wrong with the following result:

$$\lim_{x\to 1} \frac{2x^2 - x - 1}{3x^2 - 5x + 2} = \lim_{x\to 1} \frac{4x - 1}{6x - 5} = \lim_{x\to 1} \frac{4}{6} = \frac{2}{3}$$

   The problem is that $\lim\limits_{x\to 1} \dfrac{4x-1}{6x-5} = \dfrac{3}{1} = 3$ which is not an indeterminate form, so we should not have applied L'Hospital's rule another time to get the last two equalities.

3. ♣ Prove that if $r > 0$ and $x > 1$, then $\ln x \le \dfrac{x^r - 1}{r}$. Hint: Use Cauchy's form of the Mean Value Theorem with $f(x) = \ln x$ and $g(x) = x^r$.

Proof: Let $f(y) = \ln y$ and $g(y) = y^r$ for $y \in [1, x]$ for some $x > 1$. Then f and g are continuous and differentiable on $[1, x]$, so by Cauchy's form of the Mean Value Theorem there is some $c \in (1, x)$ such that $\dfrac{f(x) - f(1)}{g(x) - g(1)} = \dfrac{f'(c)}{g'(c)}$. Thus $\frac{\ln x}{x^r - 1} = \frac{\frac{1}{c}}{rc^{r-1}} = \frac{1}{rc^r} \le \frac{1}{r}$ since $c \in (1, x)$. Therefore $\ln x \le \frac{x^r - 1}{r}$.

□

4. Prove that $1 + x^2 \le e^{x^2}$ for all $x \in \mathbb{R}$.

Proof: First, if $x = 0$, clearly $1 + 0 \le e^0 = 1$, so we may assume $x \neq 0$. Let $f(y) = 1 + y^2$ and $g(y) = e^{y^2}$ for $y \in [0, x]$. (Notice that if $x < 0$, we could write our interval as $[x, 0]$.) We know f and g are continuous and differentiable on $[0, x]$, so by Cauchy's Mean Value Theorem, there is some c between 0 and x such that $\dfrac{f(x) - f(0)}{g(x) - g(0)} = \dfrac{f'(c)}{g'(c)}$, thus $\dfrac{1 + x^2 - 1}{e^{x^2} - 1} = \dfrac{2c}{e^{c^2}(2c)} = \dfrac{1}{e^{c^2}} \le \dfrac{1}{e^0} = 1$. Notice this is true for any $x \neq 0$. Therefore $x^2 \le e^{x^2} - 1 \Longrightarrow 1 + x^2 \le e^{x^2}$ for all $x \in \mathbb{R}$.

□