Math 3210-3
HW 19
Solutions

The Mean Value Theorem

1. Prove that $|\cos x - \cos y| \leq |x - y|$.

   Proof: Let $x, y \in \mathbb{R}$ and assume $x < y$. Cosine is a differentiable function on $\mathbb{R}$, so in particular it is continuous on $[x, y]$ and differentiable on (x, y). Thus by the Mean Value Theorem, there is some $c \in (x, y)$ such that $\cos'(c) = \frac{\cos x - \cos y}{x - y}$. But $\cos'(c) = \sin c$, and $|\sin c| \leq 1$, so we have $\left|\dfrac{\cos x - \cos y}{x - y}\right| \leq 1$. Therefore $|\cos x - \cos y| \leq |x - y|$.

   $\square$

2. ♣ Suppose that f is differentiable on $\mathbb{R}$ and that $f(0) = 0, f(1) = 1$ and $f(2) = 1$.

   (a) Show that $f'(x) = \frac{1}{2}$ for some $x \in (0, 2)$.

   Proof: By the Mean Value Theorem, there is some $x \in (0, 2)$ such that $f'(x) = \frac{f(2) - f(0)}{2 - 0} = \frac{1}{2}$.

   $\square$

   (b) Show that $f'(x) = \frac{1}{7}$ for some $x \in (0, 2)$.

   Proof: By the Mean Value Theorem, there is some $y \in (1, 2)$ such that $f'(y) = \frac{f(1) - f(2)}{2 - 1} = 0$. Since $f'(y) = 0 < \frac{1}{7} < \frac{1}{2} = f'(x)$ we can conclude by the Intermediate Value Theorem for Derivatives that there is some $z \in (0, 2)$ such that $f'(z) = \frac{1}{7}$.

   $\square$

3. ♣ Show that $ex \leq e^x$ for all $x \in \mathbb{R}$.

   Proof: Let $f(x) = e^x - ex$. Then $f(1) = 0$ and $f'(x) = e^x - e$. For $x < 1$ we see that $f'(x) < 0$ so f is a decreasing function on $(-\infty, 1)$. For $x > 1$ we see that $f'(x) > 0$ so f is an increasing function on $(1, \infty)$. Thus $f(x) \geq 0$ for all $x \in \mathbb{R}$ which implies that $e^x \geq ex$ for all $x \in \mathbb{R}$.

   $\square$

4. ♣ Show that $\sin x \leq x$ for all $x \geq 0$. *Hint:* Show that $f(x) = x - \sin x$ is increasing on $[0, \infty)$.

   Proof: Let $g(x) = x - \sin x$ for $x \geq 0$. Then $g(0) = 0$ and $g'(x) = 1 - \cos x \geq 0$. Thus g is always nondecreasing on $[0, \infty)$. Thus $x - \sin x \geq 0$ and $x \geq \sin x$ for $x \geq 0$.

   $\square$

5. ♣ Suppose that f and g are differentiable on $\mathbb{R}$, $f(0) = g(0)$, and $f'(x) \leq g'(x)$ for all $x \in \mathbb{R}$. Prove $f(x) \leq g(x)$ for $x \geq 0$.

   Proof: Let $h(x) = g(x) - f(x)$. Then $h(0) = 0$ and $h'(x) = g'(x) - f'(x) \geq 0$. Thus h is always nondecreasing which implies that $h(x) \geq 0$ for $x \geq 0$. Therefore $g(x) - f(x) \geq 0 \Longrightarrow g(x) \geq f(x)$ for $x \geq 0$.

   $\square$

6. Show that $f(x) = \ln x$ is uniformly continuous on $[1, \infty)$.

   Proof: We know $f(x)$ is continuous on $[1, \infty)$ and f is differentiable on $(1, \infty)$. In fact, $f'(x) = \frac{1}{x} \leq 1$ for all $x \in (1, \infty)$, so by Theorem 88, f is uniformly continuous on $[1, \infty)$.

   $\square$