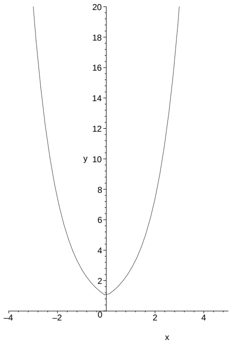
Math 3210-3
HW 18
Solutions

The Derivative

1. For each of the following functions defined on $\mathbb{R}$, give the set of points at which it is *not* differentiable. Sketches will be helpful. No proofs are required.

   (a) $e^{|x|}$

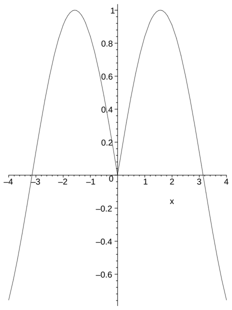
   The graph of $e^{|x|}$ is:



   The only point where this function might not be differentiable is at $x = 0$. In fact, it is not differentiable at $x = 0$ since $\lim\limits_{x\to 0^+} \dfrac{f(x) - f(0)}{x} = \lim\limits_{x\to 0^+} \dfrac{e^x - 1}{x} = \lim\limits_{x\to 0^+} \dfrac{e^x}{1} = 1$, but $\lim\limits_{x\to 0^-} \dfrac{f(x) - f(0)}{x} = \lim\limits_{x\to 0^-} \dfrac{e^{-x} - 1}{x} = \lim\limits_{x\to 0^-} \dfrac{-e^{-x}}{1} = -1$. Thus the function is not differentiable at $x = 0$.

   (b) $\sin |x|$

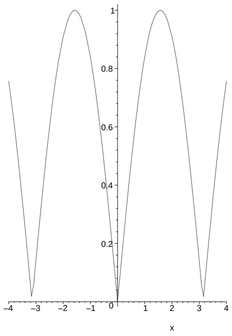
   The graph of $\sin |x|$ is:



   We can see that this function is not differentiable at $x = 0$.

   (c) $|\sin x|$

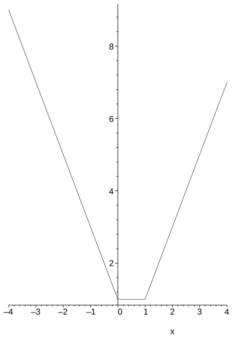
   The graph of $|\sin x|$ is:



We can see that this function is not differentiable at $x = n\pi$ where $n \in \mathbb{Z}$.

(d) $|x| + |x-1|$

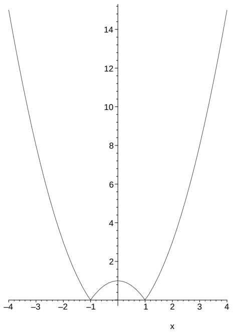
The graph of $|x| + |x-1|$ is:



We can see that this function is not differentiable at $x = 0$ and $x = 1$.

(e) $|x^2 - 1|$

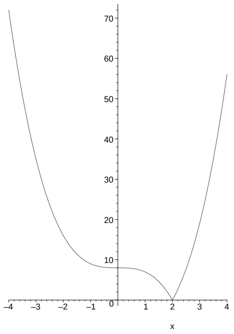
The graph of $|x^2 - 1|$ is:



We can see that this function is not differentiable at $x = -1$ and $x = 1$.

(f) $|x^3 - 8|$

The graph of $|x^3 - 8|$ is:



We can see that this function is not differentiable at $x = 2$.

2. Let $f(x) = x^2 \sin\left(\dfrac{1}{x}\right)$ for $x \neq 0$ and $f(0) = 0$.

(a) Use Theorems 81 and 82 to show that f is differentiable at each $a \neq 0$ and calculate $f'(a)$. Use, without proof, the fact that $\sin x$ is differentiable and that $\cos x$ is its derivative.

Proof: We have shown that x^2 and $\frac{1}{x}$ are differentiable for $x \neq 0$, and assuming $\sin x$ is differentiable, by Theorem 81, the sum and product of differentiable functions is differentiable, so f is differentiable for $x \neq 0$. We can use the product rule and chain rule to compute the derivative:

$$f'(a) = 2a \sin\left(\frac{1}{a}\right) + a^2 \cos\left(\frac{1}{a}\right)\left(\frac{-1}{a^2}\right)$$

□

(b) ♣ Use the definition to show that f is differentiable at $x = 0$ and that $f'(0) = 0$.

Proof: We need to show $\lim_{x\to 0} \dfrac{f(x) - f(0)}{x - 0}$ exists and is finite. We have the following:

$$\begin{aligned} \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} &= \lim_{x\to 0} \frac{x^2 \sin\left(\frac{1}{x}\right)}{x} \\ &= \lim_{x\to 0} x \sin\left(\frac{1}{x}\right) \end{aligned}$$

Now notice that $-|x| \leq x \sin\left(\frac{1}{x}\right) \leq |x|$. Also $\lim_{x\to 0} -|x| = \lim_{x\to 0} |x| = 0$, so by the squeeze theorem, $\lim_{x\to 0} x \sin\left(\dfrac{1}{x}\right) = 0$. Therefore f is differentiable at $x = 0$ and $f'(0) = \lim_{x\to 0} \dfrac{f(x)}{x} = 0$.

□

(c) Show that f' is not continuous at $x = 0$.

Proof: We will use Theorem 70 and show there is a sequence (x_n) in $\mathbb{R}$ which converges to 0, but $(f'(x_n))$ does not converge to $f'(0)$. Let $x_n = \dfrac{1}{2\pi n}$ for $n \in \mathbb{N}$. Then $(x_n) \to 0$, but $f'(x_n) = 2\left(\frac{1}{2\pi n}\right)\sin(2\pi n) - \cos(2\pi n) = -1$ for all n. Thus the sequence $(f'(x_n)) = (-1, -1, -1, -1, -1, -1, \ldots)$ which converges to $-1 \neq f'(0) = 0$. Therefore f' is not continuous at $x = 0$.

□

3. Let $f(x) = x \sin\left(\dfrac{1}{x}\right)$ for $x \neq 0$, and $f(0) = 0$.

   (a) Observe that f is continuous at $x = 0$.

   We showed in problem 2(b) that $\lim_{x\to 0} f(x) = f(0)$, so f is continuous at $x = 0$.

   (b) ♣ Is f differentiable at $x = 0$? Justify your answer.

   Notice that $\lim_{x\to 0} \dfrac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \dfrac{x \sin\left(\frac{1}{x}\right)}{x} = \lim_{x\to 0} \sin\left(\dfrac{1}{x}\right)$ which does not exist. Therefore f is not differentiable at $x = 0$.

4. Let $f(x) = x^2$ for x rational and $f(x) = 0$ for x irrational.

   (a) Prove that f is continuous at $x = 0$.

   Proof: We see that $0 \leq f(x) \leq x^2$, so by the squeeze theorem $\lim_{x\to 0} f(x) = 0 = f(0)$. Therefore f is continuous at $x = 0$.

   □

   (b) Prove that f is discontinuous at all $x \neq 0$.

   Proof: Let $x \in \mathbb{R} \setminus \{0\}$. If $x \in \mathbb{Q}$, let (s_n) be a sequence of irrationals converging to x. Then $\lim_{n\to\infty} f(s_n) = 0 \neq f(x) = x^2$. Thus f is not continuous at rational points of $\mathbb{R} \setminus \{0\}$. On the other hand, if x is irrational, let (s_n) be a sequence of rationals converging to x with $s_n > x$ if $x > 0$ and $s_n < x$ if $x < 0$. In either case we have $|s_n| > |x|$ and $s_n^2 > x^2$. This implies that $\lim_{n\to\infty} f(s_n) = \lim_{n\to\infty} s_n^2 > x^2 > 0 = f(x)$. Thus f is discontinuous at all $x \neq 0$.

   □

   (c) ♣ Prove that f is differentiable at $x = 0$. *Warning:* You cannot simply claim $f'(x) = 2x$.

   Proof: We need to show $\lim_{x\to 0} \dfrac{f(x) - f(0)}{x - 0}$ exists and is finite. We have the following:

$$\begin{aligned} \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} &= \lim_{x\to 0} \frac{f(x)}{x} \\ &= \lim_{x\to 0} g(x) \end{aligned}$$

where $g(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin Q \end{cases}$. We have seen this function in an example in class, and we proved the limit as x goes to 0 is 0. Therefore f is differentiable at $x = 0$.

$\square$