Math 3210-3
HW 17
Solutions

Uniform Continuity

1. Which of the following continuous functions are uniformly continuous on the specified set? Justify your answers. Use any theorems from this section that you wish.

   (a) $f(x) = x^{17} \sin x - e^x \cos 3x$ on $[0, \pi]$
   f is the product and sum of continuous functions, so by Theorem 71 f is continuous on a compact set, so by Theorem 76, f is uniformly continuous on $[0, \pi]$.

   (b) $f(x) = x^3$ on $[0, 1]$
   Once again, f is continuous on a compact set, so by Theorem 76, f is uniformly continuous on $[0, 1]$.

   (c) $f(x) = x^3$ on $(0, 1)$
   Let $\tilde{f} = x^3$ on $[0, 1]$. Then $\tilde{f}$ is an extension of f which is continuous, so by Theorem 78, f is uniformly continuous on $(0, 1)$.

   (d) $f(x) = x^3$ on $\mathbb{R}$
   f is not uniformly continuous on $\mathbb{R}$. We could prove this, but it is the same proof as we mentioned in class with $g(x) = x^2$. The idea is that the function values get increasingly far apart as x and y get big, even if x and y are "close" enough.

   (e) $f(x) = \frac{1}{x^3}$ on $(0, 1]$
   This function is not uniformly continuous on $(0, 1]$ since it can not be extended to a continuous funtion on $[0, 1]$ (Theorem 78). Any value we might choose for $f(0)$ would give us a function that is not continuous at 0.

2. (a) Prove that if f is uniformly continuous on a bounded set S, then f is a bounded function on S.
   Proof: If S is bounded, then $\operatorname{cl} S$ is also bounded. Since f is uniformly continuous on S, its extension $\tilde{f}$ must be continuous on $\operatorname{cl} S$. But $\operatorname{cl} S$ is also a compact set, so $\tilde{f}$ is continuous on a compact set. Theorem 73 implies its image is a compact set. By the Heine-Borel Theorem, its image is bounded. Therefore f is bounded on S.

   □

   (b) Use part (a) to prove that $\frac{1}{x^2}$ is not uniformly continuous on $(0, 1)$.
   Proof: The function $f(x) = \frac{1}{x^2}$ is not bounded on $(0, 1)$, so by the contrapositive of part (a), f is not uniformly continuous on $(0, 1)$.

   □

3. Let $g(x) = x^2 \sin\left(\frac{1}{x}\right)$ for $x \neq 0$ and $g(0) = 0$.

   (a) Prove that $g(x)$ is continuous on $\mathbb{R}$.
   Proof: Let $h(x) = x^2, j(x) = \sin x$, and $k(x) = \frac{1}{x}$. Then for $x \neq 0$, h, j, and k are continuous functions. Thus by Theorems 71 and 72, the product and composition are continuous, so $h \cdot j \circ k = g$ is continuous for $x \neq 0$. Now let $f(x) = -x^2$ and $l(x) = x^2$. Then $f(x) \leq g(x) \leq l(x)$ for all $x \in \mathbb{R}$, and $\lim_{x \to 0} f(x) = \lim_{x \to 0} l(x) = 0$, so by the squeeze theorem, $\lim_{x \to 0} g(x) = 0$, so g is continuous at 0 also. Thus g is continuous on $\mathbb{R}$.

   □

   (b) Why is g uniformly continuous on any bounded subset of $\mathbb{R}$?
   Proof: Let S be any bounded subset of $\mathbb{R}$. Then $\operatorname{cl} S$ is a closed and bounded set, and we can extend g to a function $\tilde{g}$ which is continuous on $\operatorname{cl} S$. Thus by Theorem 76 g is uniformly continuous on S.

□

(c) Is g uniformly continuous on $\mathbb{R}$?

Proof: I claim g is uniformly continuous on $\mathbb{R}$. We know g is uniformly continuous on any bounded subset of $\mathbb{R}$, so given $\epsilon > 0$, there exists $\delta_1 > 0$ such that for $x \in [-10000, 10000]$, $|x - y| < \delta_1 \implies |g(x) - g(y)| < \epsilon$. Let $\delta = \min\{\delta_1, \frac{\epsilon}{2}\}$. Now notice that for $|x| > 10000$, $\sin\left(\dfrac{1}{x}\right) \sim \dfrac{1}{x}$. Thus $|g(x) - g(y)| = |x^2 \sin\left(\frac{1}{x}\right) - y^2 \sin\left(\frac{1}{y}\right)| \sim |\frac{x^2}{x} - \frac{y^2}{y}| = |x - y| < \delta < \epsilon$. Therefore g is uniformly continuous on $\mathbb{R}$.

□