Math 3210-3
HW 16
Solutions

Properties of Continuous Functions

1. Show that $2^x = 3x$ for some $x \in (0, 1)$.

   Proof: First I claim that $f(x) = 2^x$ is a continuous function on $[0, 1]$. Let's assume e^x and $\ln x$ are continuous functions on $\mathbb{R}$. Then consider $y = \ln 2^x = x \ln 2$. This is a polynomial function since it is simply a constant $(\ln 2)$ times x. We already proved polynomial functions are continuous, so y is continuous. Now notice that $2^x = e^{\ln 2^x} = e^{x \ln 2} = e^y$. Since the composition of continuous functions is continuous, 2^x is continuous.

   Now for the proof of the problem. Let $g(x) : \mathbb{R} \to \mathbb{R}$ be defined as $g(x) = 2^x - 3x$. We have already shown that polynomial functions are continuous, so By Theorem 73, g is continuous. Also notice that $g(0) = 1 > 0$ and $g(1) = 2 - 3 = -1$, so by Lemma 4, there is some $c \in (0, 1)$ such that $g(c) = 0$. This implies that $2^c = 3c$ for some $c \in (0, 1)$.

   □

2. Suppose that $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ are continuous functions such that $f(a) \leq g(a)$ and $f(b) \geq g(b)$. Prove that $f(c) = g(c)$ for some $c \in [a, b]$.

   Proof: Let $h(x) : [a, b] \to \mathbb{R}$ be defined by $h(x) = f(x) - g(x)$. By Theorem 73, h is continuous. Also $h(a) = f(a) - g(a) \leq 0$ since $f(a) \leq g(a)$, and $h(b) = f(b) - g(b) \geq 0$ since $f(b) \geq g(b)$. If $h(a) = 0$, then we have $f(a) = g(a)$. Likewise, if $h(b) = 0$ we have $f(b) = g(b)$. If $h(a), h(b) \neq 0$, then by Lemma 4, there is some $c \in (a, b)$ such that $h(c) = 0$ which implies $f(c) = g(c)$.

   □

3. Suppose that f is a real-valued continuous function on $\mathbb{R}$ and that $f(a)f(b) < 0$ for some $a, b \in \mathbb{R}$. Prove that there exists c between a and b such that $f(c) = 0$.

   Proof: If $f(a)f(b) < 0$, then we have the following two cases: Either $f(a) > 0$ and $f(b) < 0$ or $f(a) < 0$ and $f(b) > 0$. In either case, we can apply Lemma 4 to find a value $c \in (a, b)$ such that $f(c) = 0$.

   □