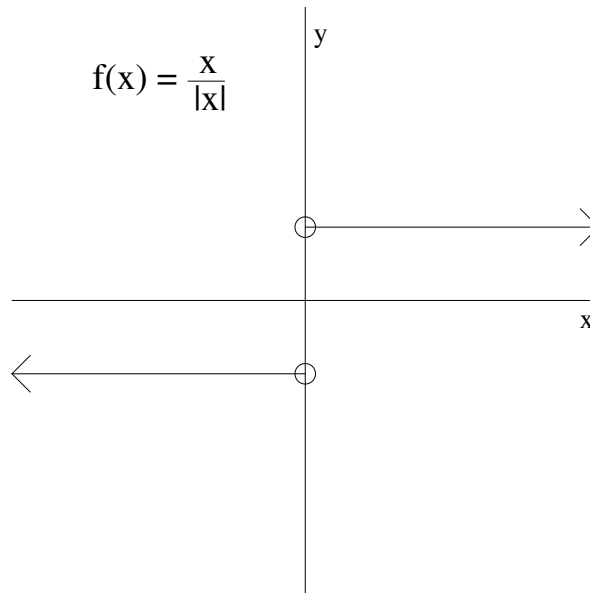


Math 3210-3
HW 15
Solutions

Limits of Functions

1. Sketch the function $f(x) = \frac{x}{|x|}$. Determine, by inspection, the limits $\lim_{x \rightarrow \infty} f(x)$, $\lim_{x \rightarrow 0^+} f(x)$, $\lim_{x \rightarrow 0^-} f(x)$, $\lim_{x \rightarrow -\infty} f(x)$, and $\lim_{x \rightarrow 0} f(x)$ when they exist. Also indicate when they do not exist.



$$\begin{aligned}\lim_{x \rightarrow \infty} f(x) &= 1 \\ \lim_{x \rightarrow 0^+} f(x) &= 1 \\ \lim_{x \rightarrow 0^-} f(x) &= -1 \\ \lim_{x \rightarrow -\infty} f(x) &= -1 \\ \lim_{x \rightarrow 0} f(x) &\text{ does not exist}\end{aligned}$$

2. Find the following limits and prove your answers.

(a) ♣ $\lim_{x \rightarrow 0} |x|$

Proof: I claim $\lim_{x \rightarrow 0} |x| = 0$. Let $\epsilon > 0$. Let $\delta = \epsilon$. Then when $|x - 0| = |x| < \delta$, we have $||x| - 0| = |x| < \delta = \epsilon$. Therefore $\lim_{x \rightarrow 0} |x| = 0$.

□

(b) $\lim_{x \rightarrow 0} \frac{x^2}{|x|}$

Proof: I claim $\lim_{x \rightarrow 0} \frac{x^2}{|x|} = 0$. Let $\epsilon > 0$, and let $\delta = \epsilon$. Then when $|x - 0| = |x| < \delta$, we have

$$\begin{aligned} \left| \frac{x^2}{|x|} - 0 \right| &= \left| \frac{x^2}{|x|} \right| \\ &= \frac{x^2}{|x|} \\ &= \begin{cases} \frac{x^2}{x} & \text{when } x > 0 \\ \frac{x^2}{-x} & \text{when } x < 0 \end{cases} \\ &= \begin{cases} x & \text{when } x > 0 \\ -x & \text{when } x < 0 \end{cases} \\ &= |x| \\ &< \delta \\ &= \epsilon \end{aligned}$$

Therefore $\lim_{x \rightarrow 0} \frac{x^2}{|x|}$.

□

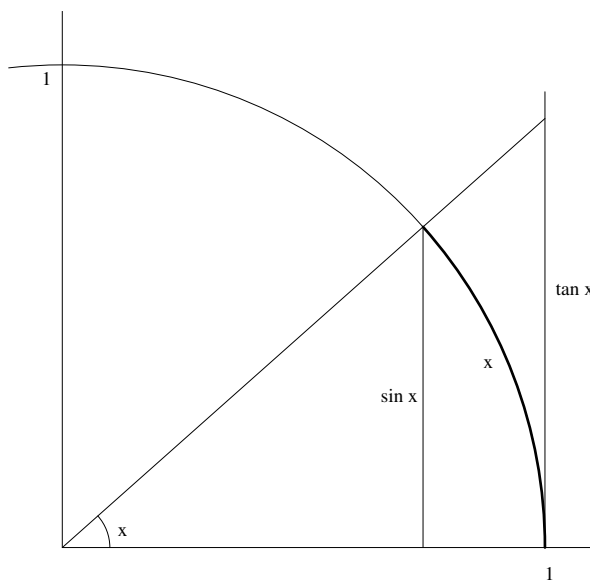
3. Let $f(x) = \frac{\sin x}{|x|}$ for $x \neq 0$. Find $\lim_{x \rightarrow 0^+} f(x)$ and $\lim_{x \rightarrow 0^-} f(x)$. Does $\lim_{x \rightarrow 0} f(x)$ exist?

Proof: For $x \in \left(0, \frac{\pi}{2}\right)$, we know $\sin x \leq x \leq \tan x = \frac{\sin x}{\cos x}$. (See the figure below.) Thus $\frac{\cos x}{\sin x} \leq \frac{1}{x} \leq \frac{1}{\sin x} \implies \cos x \leq \frac{\sin x}{x} \leq 1$. Hence we have $\lim_{x \rightarrow 0^+} \cos x = \lim_{x \rightarrow 0^+} 1 = 1$, so by the squeeze theorem, $\lim_{x \rightarrow 0^+} f(x) = 1$.

On the other hand, if $x \in \left(-\frac{\pi}{2}, 0\right)$, then $f(x) = \frac{\sin x}{-x}$ and we have $-\sin x \geq -x \geq -\tan x = \frac{-\sin x}{\cos x} \implies \frac{-\cos x}{\sin x} \leq \frac{-1}{x} \leq \frac{-1}{\sin x} \implies -\cos x \leq \frac{-\sin x}{x} \leq -1$. Once again $\lim_{x \rightarrow 0^-} -\cos x = \lim_{x \rightarrow 0^-} -1 = -1$, so by the squeeze theorem $\lim_{x \rightarrow 0^-} f(x) = -1$.

Since the one-sided limits are not equal, $\lim_{x \rightarrow 0} f(x)$ does not exist.

□



4. Let f, g and h be functions from D into \mathbb{R} , and let c be an accumulation point of D . If $f(x) \leq g(x) \leq h(x)$ for all $x \in D$ with $x \neq c$, and if $\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} h(x) = L$, then prove that $\lim_{x \rightarrow c} g(x) = L$.
- Proof:* Let (x_n) be a sequence in D which converges to c . Then $(f(x_n)) \rightarrow L$ and $(h(x_n)) \rightarrow L$ by Theorem 68. Also, $f(x_n) \leq g(x_n) \leq h(x_n)$ by hypothesis, so by the squeeze theorem, $(g(x_n)) \rightarrow L$. Thus from Theorem 68, $\lim_{x \rightarrow c} g(x) = L$.

□

Continuous Functions

5. ♣ Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be given by $f(x) = x^2 - 3x + 5$. Use the definition of a continuous function to prove that f is continuous at 2.

Proof: Let $\epsilon > 0$. Let $\delta = \min\{1, \frac{\epsilon}{2}\}$. Then if $|x - 2| < \delta$, then $|x - 2| < 1$, so $|x - 1| < 2$. Thus $|f(x) - f(2)| = |x^2 - 3x + 5 - (4 - 6 + 5)| = |x^2 - 3x + 2| = |x - 2||x - 1| < 2\frac{\epsilon}{2} = \epsilon$. Therefore f is continuous at 2.

□

6. ♣ Let $f : D \rightarrow \mathbb{R}$ and define $|f| : D \rightarrow \mathbb{R}$ by $|f|(x) = |f(x)|$. Suppose that f is continuous at $c \in D$. Prove that $|f|$ is continuous at c .

Proof: Since f is continuous at $c \in D$, by Theorem 71, there is some convergent sequence (s_n) in D which converges to c , and $(f(s_n)) \rightarrow f(c)$. But by HW 12 problem 6 (see below), if $(f(s_n)) \rightarrow f(c)$, then $|f(s_n)| \rightarrow |f(c)|$. This implies $|f|(s_n) \rightarrow |f|(c)$ by definition, so by Theorem 71, $|f|$ is continuous at c .

□

6. (From HW 12) Prove or give a counterexample: If (s_n) converges to s , then $(|s_n|)$ converges to $|s|$.

Proof: First we will prove a lemma:

Lemma 1

$||x| - |y|| \leq |x - y|$ for all $x, y \in \mathbb{R}$.

Proof: Suppose $x, y \geq 0$. Then $||x| - |y|| = |x - y|$. If $x \geq 0$ and $y < 0$, then $x - y > 0$, so we have $||x| - |y|| = |x - (-y)| = |x + y| \leq |x| + |y| = x - y = |x - y|$. If $x < 0$ and $y \geq 0$, $x - y < 0$, so we have $||x| - |y|| = |-x - y| = |(-1)(x + y)| = |x + y| \leq |x| + |y| = -x + y = (-1)(x - y) = |x - y|$. Finally, if $x, y < 0$, then $||x| - |y|| = |-x + y| = |(-1)(x - y)| = |x - y|$. We have covered all of the cases, so we can conclude that $||x| - |y|| \leq |x - y|$ for all $x, y \in \mathbb{R}$.

□

Now for the proof of the problem. Let $\epsilon > 0$. Then there exists N such that for all $n > N$, $|s_n - s| < \epsilon$. But by the Lemma, we have $||s_n| - |s|| \leq |s_n - s| < \epsilon$. Therefore $(|s_n|)$ converges to $|s|$.

□

7. ♣ Suppose that f satisfies $f(x + y) = f(x) + f(y)$, and that f is continuous at 0. Prove that f is continuous at a for all a .

Proof: Let D be the domain of f . First note that if $f(x + y) = f(x) + f(y)$ for all $x, y \in D$, and if f is continuous at 0, then $0 \in D$, and we have $f(0) = f(0 + 0) = f(0) + f(0) = 2f(0)$ which implies that $f(0) = 0$. Let $a \in D$, and let $\epsilon > 0$. Then there exists a $\delta > 0$ such that for all $|x| < \delta$, $|f(x)| < \epsilon$. This implies that if $|x - a| < \delta$, then $|f(x) - f(a)| = |f(x - a)| < \epsilon$. Therefore f is continuous at a for all a in its domain.

□