

**Math 3210-3**  
HW 14  
Solutions

## Subsequences

1. For each sequence, find the set  $S$  of subsequential limits, the  $\limsup$ , and the  $\liminf$ . No proofs are needed.

(a)  $w_n = \frac{(-1)^n}{n}$ :  $S = \{0\}$ ,  $\limsup w_n = \liminf w_n = 0$

(b)  $(x_n) = (0, 1, 2, 0, 1, 3, 0, 1, 4, \dots)$ :  $S = \{0, 1\}$ ,  $\limsup x_n = +\infty$ ,  $\liminf x_n = 0$

(c)  $y_n = n[2 + (-1)^n]$ :  $S = \emptyset$ ,  $\limsup y_n = +\infty$ ,  $\liminf y_n = +\infty$

(d)  $z_n = (-n)^n$ :  $S = \emptyset$ ,  $\limsup z_n = +\infty$ ,  $\liminf z_n = -\infty$

(e)  $b_n = \sin\left(\frac{n\pi}{3}\right)$ :  $S = \left\{\frac{-\sqrt{3}}{2}, 0, \frac{\sqrt{3}}{2}\right\}$ ,  $\limsup b_n = \frac{\sqrt{3}}{2}$ ,  $\liminf b_n = \frac{-\sqrt{3}}{2}$

2. If  $\limsup a_n$  and  $\limsup b_n$  are finite, prove that  $\limsup(a_n + b_n) \leq \limsup a_n + \limsup b_n$ .

*Proof:* Let  $c_n = a_n + b_n$  and let  $a = \limsup a_n$ ,  $b = \limsup b_n$  and  $c = \limsup c_n$ . Let  $\epsilon > 0$ . Then by Theorem 64, there exists  $N_1$  such that for all  $n > N_1$ , then  $a_n \leq a + \frac{\epsilon}{2}$ . Similarly, there exists  $N_2$  such that for all  $n > N_2$ ,  $b_n \leq b + \frac{\epsilon}{2}$ . Thus we have  $c_n = a_n + b_n \leq a + b + \epsilon$  for all  $n > \max\{N_1, N_2\}$ . Hence by Theorem 26,  $c_n \leq a + b$  for all  $n > \max\{N_1, N_2\}$ . Let  $(c_{n_k})$  be any subsequence in  $(c_n)$  which converges to  $c$ . For  $n_k > N$ ,  $c_{n_k} \leq a + b$ , so by Theorem 54,  $c = \lim c_{n_k} \leq a + b$ . Therefore  $\limsup c_n \leq \limsup a_n + \limsup b_n$ .

□

3. Let  $(r_n)$  be an enumeration of the set  $\mathbb{Q}$ . Show that there exists a subsequence  $(r_{n_k})$  such that  $\lim_{k \rightarrow \infty} r_{n_k} = +\infty$ .

*Proof:* Let  $A_1 = \{n \in \mathbb{N} : r_n > 0\}$ , and let  $n_1 = \min A_1$ . Let  $A_2 = \{n \in \mathbb{N} : n > n_1 \text{ and } r_n > r_{n_1} + 1\}$ , and let  $n_2 = \min A_2$ . Continue in this way to get  $n_3, n_4, \dots$ . So in general  $n_k = \min\{n \in \mathbb{N} : k > n_{k-1} \text{ and } r_n > r_{n_{k-1}} + 1\}$ . Then  $(r_{n_k})$  is an increasing sequence. Also for  $\epsilon = \frac{1}{2}$ ,  $|r_{n_p} - r_{n_m}| > \epsilon$  for all  $p, m$ , which implies that  $(r_{n_k})$  is not Cauchy, so it diverges and  $r_{n_k} > 0$  for all  $n_k$ . Thus  $\lim r_{n_k} = +\infty$ .

□