# Math 3210-3
## HW 13
Solutions

Note: Problems 4 and 7 are extra credit.

## Limit Theorems

1. Suppose that $\lim a_n = a$ and $\lim b_n = b$. Let $s_n = \dfrac{a_n^3 + 4a_n}{b_n^2 + 1}$. Prove that $\lim s_n = \dfrac{a^3+4a}{b^2+1}$ carefully, using the limit theorems.

   *Proof:* We have the following:

$$\begin{aligned}
\lim s_n &= \lim \frac{a_n^3 + 4a_n}{b_n^2+1} && \text{by definition}\\
&= \frac{\lim(a_n^3 + 4a_n)}{\lim(b_n^2+1)} && \text{by Theorem 50(4)}\\
&= \frac{\lim(a_n^3) + \lim(4a_n)}{\lim(b_n^2) + \lim 1} && \text{by Theorem 50(1)}\\
&= \frac{(\lim a_n)^3 + \lim(4a_n)}{(\lim b_n)^2 + \lim 1} && \text{by Theorem 50(3)}\\
&= \frac{(\lim a_n)^3 + 4\lim a_n}{(\lim b_n)^2 + 1} && \text{by Theorem 50(2)}\\
&= \frac{a^3+4a}{b^2+1} && \text{by hypothesis}
\end{aligned}$$

□

2. (a) Verify $1 + a + a^2 + \cdots + a^n = \dfrac{1-a^{n+1}}{1-a}$ for $a \neq 1$.

   *Proof:* We will prove this by induction. Let $n = 1$. Then $1 + a = \frac{1-a^2}{1-a} = \frac{(1+a)(1-a)}{1-a} = 1 + a$. Hence $n = 1$ is true. Assume $1 + a + a^2 + \cdot + a^k = \frac{1-a^{k+1}}{1-a}$ for some $k \in \mathbb{N}$. Then we have the following:

$$\begin{aligned}
1 + a + a^2 + \cdots + a^k + a^{k+1} &= \frac{1-a^{k+1}}{1-a} + a^{k+1}\\
&= \frac{1-a^{k+1}+a^{k+1}-a^{k+2}}{1-a}\\
&= \frac{1-a^{k+2}}{1-a}
\end{aligned}$$

   Therefore $1 + a + a^2 + \cdots + a^n = \dfrac{1-a^{n+1}}{1-a}$ for $a \neq 1$.

□

   (b) ♣ Find $\lim_{n\to\infty}(1 + a + a^2 + \cdots + a^n)$ for $|a| < 1$.

   *Proof:* By Theorem 50, we see that $\lim_{n\to\infty}(1 + a + a^2 + \cdots + a^n) = \lim_{n\to\infty} \frac{1-a^{n+1}}{1-a} = \frac{\lim(1-a^{n+1})}{\lim(1-a)} = \frac{1-\lim a^{n+1}}{1-a}$. Since $|a| < 1$, we know from problem 3(f), $\lim a^n = 0$, so $\lim_{n\to\infty}(1 + a + a^2 + \cdots + a^n) = \frac{1}{1-a}$.

□

(c) Calculate $\lim_{n\to\infty}(1+\frac{1}{3}+\frac{1}{9}+\cdots+\frac{1}{3^n})$.

By part (b), $\lim\limits_{n\to\infty}\left(1+\dfrac{1}{3}+\dfrac{1}{9}+\cdots+\dfrac{1}{3^n}\right)=\dfrac{1}{1-\frac{1}{3}}=\dfrac{3}{2}$.

(d) What is $\lim_{n\to\infty}(1+a+a^2+\cdots+a^n)$ for $a\geq 1$?

If $a\geq 1$, $\lim_{n\to\infty}(1+a+a^2+\cdots+a^n)=+\infty$. To prove this, recall that if $a\geq 1$, then $a^n>a$ for all $n\in\mathbb{N}$. So $1+a+a^2+\cdots+a^n>1+a+a+\cdots+a=1+na>n$. Therefore for any $M\in\mathbb{R}$, by the Archimedean property, there is some $n\in\mathbb{N}$ such that $n>M$, but we also have that $n<1+na<1+a+a+\cdots+a<1+a+a^2+\cdots+a^n$. Therefore, $\lim_{n\to\infty}(1+a+a^2+\cdots+a^n)=+\infty$.

## Monotone and Cauchy Sequences

3. Which of the following sequences are nondecreasing? nonincreasing? bounded? No proofs required.

   (a) $\dfrac{1}{n}$ **Nonincreasing, Bounded**

   (b) $\dfrac{(-1)^n}{n^2}$ **Bounded**

   (c) $n^5$ **Nondecreasing**

   (d) $\sin\left(\dfrac{n\pi}{7}\right)$ **Bounded**

   (e) $(-2)^n$ **Neither**

   (f) $\dfrac{n}{3^n}$ **Nonincreasing, Bounded**

4. Let $(s_n)$ be a nondecreasing sequence of positive numbers and define $\sigma_n=\dfrac{s_1+s_2+\cdots+s_n}{n}$. Prove that $(\sigma_n)$ is a nondecreasing sequence.

*Proof:* We will prove this by induction. We know that $0<s_1\leq s_2$, so $s_1+s_1\leq s_1+s_2$, which implies that $2s_1\leq s_1+s_2$. Thus $s_1\leq\dfrac{s_1+s_2}{2}$. So $n=1$ is true. Assume $\sigma_k$ is true, that is $\frac{s_1+s_2+\cdots+s_k}{k}\leq\frac{s_1+s_2+\cdots+s_k+s_{k+1}}{k+1}$. By hypothesis, we know $s_n\geq s_j$ for all $j<n$, so $s_1+s_2+\cdots+s_{k+1}\leq s_{k+1}+s_{k+1}+\cdots+s_{k+1}$ where there are $k+1$ summands. So $s_1+s_2+\cdots+s_{k+1}\leq s_{k+1}(k+1)$. Thus $0\leq s_{k+2}(k+1)-s_1-s_2-\cdots-s_{k+1}$. Hence we have the following:

$$\begin{aligned}
\frac{s_1+\cdots+s_{k+1}}{k+1} &\leq \frac{s_1+\cdots+s_{k+1}}{k+1}+\frac{s_{k+2}(k+1)-s_1-s_2-\cdots-s_{k+1}}{(k+2)(k+1)}\\
&= \frac{(k+2)(s_1+\cdots+s_{k+1})+s_{k+2}(k+1)-s_1-s_2-\cdots-s_{k+1}}{(k+2)(k+1)}\\
&= \frac{s_1k+\cdots+s_{k+1}k+2s_1+\cdots+2s_{k+1}+s_{k+2}(k+1)-s_1-s_2-\cdots-s_{k+1}}{(k+2)(k+1)}\\
&= \frac{s_1k+s_2k+\cdots+s_{k+1}k+s_1+s_2+\cdots+s_{k+1}+s_{k+2}(k+1)}{(k+2)(k+1)}\\
&= \frac{s_1(k+1)+s_2(k+1)+\cdots+s_{k+1}(k+1)+s_{k+2}(k+1)}{(k+1)(k+2)}\\
&= \frac{s_1+s_2+\cdots+s_{k+2}}{k+2}\\
&= \sigma_{k+2}
\end{aligned}$$

Therefore $(\sigma_n)$ is a nondecreasing sequence.

□

5. ♣ Let $s_1=1$ and $s_{n+1}=\dfrac{s_n+1}{3}$ for $n\geq 1$.

   (a) Find $s_2, s_3$ and $s_4$.

   $s_2=\frac{2}{3}$, $s_3=\frac{5}{9}$, and $s_4=\frac{14}{27}$.

(b) Use induction to show that $s_n > \frac{1}{2}$ for all $n \in \mathbb{N}$.

*Proof:* We can see from part (a) that $s_1$ and $s_2$ are both $> \frac{1}{2}$. Now assume $s_k > \frac{1}{2}$ for some $k \in \mathbb{N}$. Then we have $s_k = \frac{s_{k-1}+1}{3} > \frac{1}{2}$, so

$$\begin{aligned} \frac{\frac{s_{k-1}+1}{3}+1}{3} &> \frac{\frac{1}{2}+1}{3} \\ &= \frac{\frac{3}{2}}{3} \\ &= \frac{1}{2} \end{aligned}$$

Thus $s_n > \frac{1}{2}$ for all $n \in \mathbb{N}$.

$\square$

(c) Show that $(s_n)$ is a nonincreasing sequence.

*Proof:* We need to show that $s_n \geq s_{n+1}$ for all $n \in \mathbb{N}$. From above, we see that $s_1 = 1 > \frac{2}{3} = s_2$, so $n = 1$ is true. Assume $s_k > s_{k+1}$ for some $k \in \mathbb{N}$. Then $s_{k+1} = \frac{s_k+1}{3} \geq \frac{s_{k+1}+1}{3} = s_{k+2}$. Therefore $s_n$ is a nonincreasing sequence.

$\square$

(d) Show that $\lim s_n$ exists and find $\lim s_n$.

*Proof:* Since $0 < s_n < 1$ for all $n \in \mathbb{N}$, and since $(s_n)$ is monotonic, by Theorem 59, $(s_n)$ converges to a real number, so the limit exists. We know $\lim s_n = \lim s_{n+1}$, so if $s = \lim s_n$, we have

$$\begin{aligned} s &= \lim s_n \\ &= \lim \frac{s_n+1}{3} \\ &= \frac{\lim s_n + 1}{3} \\ &= \frac{s+1}{3} \end{aligned}$$

Thus $s = \frac{s+1}{3}$. We can solve for $s$ to get $s = \frac{1}{2}$.

$\square$

6. $\clubsuit$ Let $s_n = a_1 + a_2 + \cdots + a_n$ where each $a_i \in \mathbb{R}$, and let $t_n = |a_1| + |a_2| + \cdots + |a_n|$. Prove that if $(t_n)$ is a bounded sequence then $(s_n)$ converges.

*Proof:* Notice that we have:

$$\begin{aligned} t_1 &= |a_1| \\ t_2 &= |a_1| + |a_2| \\ t_3 &= |a_1| + |a_2| + |a_3| \\ &\vdots \\ t_n &= |a_1| + |a_2| + |a_3| + \cdots + |a_n| \\ &\vdots \end{aligned}$$

So $(t_n)$ is a nondecreasing sequence. By assumption it is bounded, so by the Monotone Convergence Theorem $(t_n)$ converges. Thus $(t_n)$ is Cauchy. So by definition given $\epsilon > 0$ there exists $N$ such that for all $n \geq m > N$, $|t_n - t_m| < \epsilon$. But

$$\begin{aligned} |t_n - t_m| &= ||a_1| + |a_2| + \cdots + |a_n| - (|a_1| + |a_2| + \cdots + |a_n| + |a_{n+1}| + \cdots + |a_m|| \\ &= ||a_{n+1}| + |a_{n+2}| + \cdots + |a_m|| \\ &< \epsilon \end{aligned}$$

Thus for this same $N$ value, if $n \geq m > N$ we have

$$\begin{aligned} |s_n - s_m| &= |a_1 + \cdots + a_n - (a_1 + \cdots + a_m)| \\ &= |a_{n+1} + a_{n+2} + \cdots + a_m| \\ &\leq |a_{n+1}| + |a_{n+2}| + \cdots + |a_m| \\ &= ||a_{n+1}| + |a_{n+2}| + \cdots + |a_m|| \\ &< \epsilon \end{aligned}$$

Therefore $|s_n - s_m| < \epsilon$, so $(s_n)$ is Cauchy.

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7. If $|a_{n+1} - a_n| < 3^{-n}$ for all $n \in \mathbb{N}$, prove that $(a_n)$ is Cauchy, and then conclude that $(a_n)$ converges. (At some point in your proof problem 2(c) could be helpful.)

*Proof:* Let $\epsilon > 0$ and let $N = \dfrac{-\ln\left(\frac{2\epsilon}{3}\right)}{\ln 3}$. Then for all $n \geq m > N$ we have

$$\begin{aligned} |a_n - a_m| &= |a_n - a_{n-1} + a_{n-1} - a_{n-2} + a_{n-2} - \cdots + a_{m+1} - a_m| \\ &\leq |a_n - a_{n-1}| + |a_{n-1} - a_{n-2}| + \cdots + |a_{m+1} - a_m| \\ &< 3^{-(n-1)} + 3^{-(n-2)} + \cdots + 3^{-m} \end{aligned}$$

Since $n \geq m$ let $k \in \mathbb{N}$ with $n = m + k$. Then

$$\begin{aligned} |a_n - a_m| &< 3^{-m}(3^{-(k-1)} + 3^{-(k-2)} + \cdots + 3^{-1} + 1) \\ &< 3^{-m}\left[\sum_{i=0}^{\infty}\left(\frac{1}{3}\right)^i\right] \\ &= 3^{-m} \cdot \frac{3}{2} \\ &= \left(\frac{3}{2}\right)3^{-m} \end{aligned}$$

But $m > N$, so $|a_n - a_m| < \left(\dfrac{3}{2}\right)3^{-N} = \epsilon$. Therefore $(a_n)$ is Cauchy, and since every Cauchy sequence in the reals converges, $(a_n)$ converges.

☐

Note: $\left(\frac{3}{2}\right)3^{-N} = \epsilon \iff 3^{-N} = \frac{2\epsilon}{3} \iff -N\ln 3 = \ln\frac{2\epsilon}{3} \iff N = \dfrac{-\ln\frac{2\epsilon}{3}}{\ln 3}$.

8. Find an example of a sequence of real numbers satisfying each set of properties.
   (a) Cauchy, but not monotone: Let $a_n = \frac{(-1)^n}{n}$
   (b) Monotone, but not Cauchy: Let $a_n = n$
   (c) Bounded, but not Cauchy: Let $a_n = \begin{cases} \frac{1}{n} & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even} \end{cases}$