

# Math 3210-3

## HW 13

Due Tuesday, October 16, 2007

Note: Problems 4 and 7 are extra credit.

### Limit Theorems

1. Suppose that  $\lim a_n = a$  and  $\lim b_n = b$ . Let  $s_n = \frac{a_n^3 + 4a_n}{b_n^2 + 1}$ . Prove that  $\lim s_n = \frac{a^3 + 4a}{b^2 + 1}$  carefully, using the limit theorems.
2. (a) Verify  $1 + a + a^2 + \cdots + a^n = \frac{1 - a^{n+1}}{1 - a}$  for  $a \neq 1$ .  
(b) ♣ Find  $\lim_{n \rightarrow \infty} (1 + a + a^2 + \cdots + a^n)$  for  $|a| < 1$ .  
(c) Calculate  $\lim_{n \rightarrow \infty} (1 + \frac{1}{3} + \frac{1}{9} + \cdots + \frac{1}{3^n})$ .  
(d) What is  $\lim_{n \rightarrow \infty} (1 + a + a^2 + \cdots + a^n)$  for  $a \geq 1$ ?

### Monotone and Cauchy Sequences

3. Which of the following sequences are nondecreasing? nonincreasing? bounded? No proofs required.
  - (a)  $\frac{1}{n}$
  - (b)  $\frac{(-1)^n}{n^2}$
  - (c)  $n^5$
  - (d)  $\sin\left(\frac{n\pi}{7}\right)$
  - (e)  $(-2)^n$
  - (f)  $\frac{n}{3^n}$
4. Let  $(s_n)$  be a nondecreasing sequence of positive numbers and define  $\sigma_n = \frac{s_1 + s_2 + \cdots + s_n}{n}$ . Prove that  $(\sigma_n)$  is a nondecreasing sequence.
5. ♣ Let  $s_1 = 1$  and  $s_{n+1} = \frac{s_n + 1}{3}$  for  $n \geq 1$ .
  - (a) Find  $s_2, s_3$  and  $s_4$ .
  - (b) Use induction to show that  $s_n > \frac{1}{2}$  for all  $n \in \mathbb{N}$ .
  - (c) Show that  $(s_n)$  is a nonincreasing sequence.
  - (d) Show that  $\lim s_n$  exists and find  $\lim s_n$ .
6. ♣ Let  $s_n = a_1 + a_2 + \cdots + a_n$  where each  $a_i \in \mathbb{R}$ , and let  $t_n = |a_1| + |a_2| + \cdots + |a_n|$ . Prove that if  $(t_n)$  is a bounded sequence then  $(s_n)$  converges.
7. If  $|a_{n+1} - a_n| < 3^{-n}$  for all  $n \in \mathbb{N}$ , prove that  $(a_n)$  is Cauchy, and then conclude that  $(a_n)$  converges. (At some point in your proof problem 2(c) could be helpful.)
8. Find an example of a sequence of real numbers satisfying each set of properties.
  - (a) Cauchy, but not monotone
  - (b) Monotone, but not Cauchy
  - (c) Bounded, but not Cauchy