NOTE: Only turn in problems 1(d), 2(a), 2(c), 3(c), 4(a), and 7.

Sequences

1. Write out the first seven terms of each sequence.

(a) \( a_n = n^2 \)
   
   1, 4, 9, 16, 25, 36, 49, . . .

(b) \( b_n = \frac{(-1)^n}{n} \)
   
   \(-1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \ldots\)

(c) \( c_n = \cos \frac{2\pi}{3n} \)
   
   \(\frac{1}{2}, -\frac{1}{2}, -1, \frac{1}{2}, 1, \frac{1}{2}, \ldots\)

(d) \( d_n = \frac{2n + 1}{3n - 1} \)

   \(\frac{3}{2}, 1, \frac{5}{2}, -\frac{11}{12}, -\frac{1}{2}, 3/4, \ldots\)

2. Using only the definition of a limit of a sequence, prove the following.

(a) For any real number \( k \), \( \lim_{n \to \infty} \left( \frac{k}{n} \right) = 0. \)

   \( \text{Proof: } \) Let \( \epsilon > 0 \) and let \( N = \frac{|k|}{\epsilon} \). Then for all \( n > N \), \( \left| \frac{k}{n} - 0 \right| = \frac{|k|}{n} < \frac{|k|}{N} = \epsilon. \) Therefore \( \lim_{n \to \infty} \left( \frac{k}{n} \right) = 0. \)

(b) \( \blacklozenge \) For any real number \( k > 0 \), \( \lim_{n \to \infty} \left( \frac{1}{n^k} \right) = 0. \)

   \( \text{Proof: } \) Let \( \epsilon > 0 \) and let \( N = \frac{1}{\epsilon^{1/k}} \). Then for all \( n > N \), we have \( \left| \frac{1}{n^k} - 0 \right| = \frac{1}{n^k} < \frac{1}{N^k} = \epsilon. \)

(c) \( \lim_{n \to \infty} \frac{3n + 1}{n + 2} = 3. \)

   \( \text{Proof: } \) Let \( \epsilon > 0 \) and \( N = \frac{5}{\epsilon} \). Then for all \( n > N \), \( \left| \frac{3n + 1}{n + 2} - 3 \right| = \left| \frac{-5}{n + 2} \right| = \frac{5}{n + 2} < \frac{5}{N} = \epsilon. \) Therefore \( \lim_{n \to \infty} \frac{3n + 1}{n + 2} = 3. \)

(d) \( \blacklozenge \) \( \lim_{n \to \infty} \frac{\sin n}{n} = 0. \)

   \( \text{Proof: } \) Let \( \epsilon > 0 \) and let \( N = \frac{1}{\epsilon} \). Then for all \( n > N \), \( \left| \frac{\sin n}{n} - 0 \right| = \left| \frac{\sin n}{n} \right| \leq \frac{1}{n} < \frac{1}{N} = \epsilon. \) Therefore \( \lim_{n \to \infty} \frac{\sin n}{n} = 0. \)
3. Using any of the Theorems 47-49 or the examples we worked in class from section 4.1, prove the following.

(a) \( \lim_{n \to \infty} \frac{1}{1 + 3n} = 0 \).

\[ \text{Proof:} \quad \text{Notice that } \left| \frac{1}{1 + 3n} - 0 \right| = \frac{1}{1 + 3n} < \frac{1}{n}. \text{ And } \lim_{n \to \infty} \frac{1}{n} = 0, \text{ so by Theorem 47, } \lim_{n \to \infty} \frac{1}{1 + 3n} = 0. \]

(b) \( \lim_{n \to \infty} \frac{4n^2 - 7}{2n^3 - 5} = 0 \).

\[ \text{Proof:} \quad \text{If } n \geq 2, \text{ we have } |4n^2 - 7| = 4n^2 - 7 < 4n^2, \text{ and } |2n^3 - 5| = 2n^3 - 5 > n^3. \text{ Therefore } \frac{4n^2 - 7}{2n^3 - 5} < \frac{4n^2}{n^3} = 4 \frac{1}{n}. \text{ Since } \lim_{n \to \infty} \frac{1}{n} = 0, \text{ by Theorem 47, we can conclude that } \lim_{n \to \infty} \frac{4n^2 - 7}{2n^3 - 5} = 0. \]

(c) \( \lim_{n \to \infty} \frac{6n^2 + 5}{2n^2 - 3n} = 3 \).

\[ \text{Proof:} \quad \text{If } n > 5, \text{ we have } \left| \frac{6n^2 + 5}{2n^2 - 3n} - 3 \right| = \left| \frac{9n + 5}{2n^2 - 3n} \right| < \frac{10n}{n^2} = \frac{10}{n} = 10 \frac{1}{n}. \text{ Also we proved in class that } \lim_{n \to \infty} \frac{1}{n} = 0, \text{ so by Theorem 47, } \lim_{n \to \infty} \frac{6n^2 + 5}{2n^2 - 3n} = 3. \]

(d) \( \lim_{n \to \infty} \frac{\sqrt{n}}{n + 1} = 0. \)

\[ \text{Proof:} \quad \text{Notice that } \left| \frac{\sqrt{n}}{n + 1} - 0 \right| = \frac{\sqrt{n}}{n + 1} < \frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}}. \text{ We proved in class that } \lim_{n \to \infty} \frac{1}{\sqrt{n}} = 0, \text{ so once again by Theorem 47, we have } \lim_{n \to \infty} \frac{\sqrt{n}}{n + 1} = 0. \]

(e) \( \lim_{n \to \infty} \frac{n^2}{n!} = 0. \)

\[ \text{Proof:} \quad \text{I claim that } n^3 < n! \text{ for } n \geq 6. \text{ Let } n = 6. \text{ Then } 6^3 = 216 < 720 = 6!. \text{ Hence the base case is true. Assume that } k^3 < k! \text{ for some } k \geq 6. \text{ Then } (k+1)^3 = k^3 + 3k^2 + 3k + 1 < k! + 3k^2 + 3k + 1 < k! + k! + k! + \cdots + k! \text{ where there are } k + 1 \text{ summands. Then } (k+1)^3 < (k+1)k! = (k+1)!. \text{ Therefore } n^3 < n! \text{ for all } n \geq 6. \text{ So we now have } \left| \frac{\frac{n^2}{n!}}{\frac{n^2}{n^3}} \right| = \frac{n^2}{n!} < \frac{n^2}{n^3} = \frac{1}{n}. \text{ We proved in class that } \lim_{n \to \infty} \frac{1}{n} = 0, \text{ so by Theorem 47, } \lim_{n \to \infty} \frac{n^2}{n!} = 0. \]

(f) If \( |x| < 1 \), then \( \lim_{n \to \infty} x^n = 0. \)

\[ \text{Proof:} \quad \text{First, if } x = 0, \text{ then clearly } \lim_{n \to \infty} x^n = 0. \text{ So suppose } x \neq 0. \text{ Then } |x| < 1, \text{ so there is some } y \in \mathbb{R} \text{ such that } |x| = \frac{1}{1+y}. \text{ I claim that } (1+y)^n \geq 1 + ny \text{ for all } n \in \mathbb{N}. \text{ I will prove this claim by induction. If } n = 1, (1+y) = 1 + 1y, \text{ so the claim is true for } n = 1. \text{ Now suppose } (1+y)^k \geq 1 + ky \text{ for some } k \in \mathbb{N}. \text{ Then } (1+y)^{k+1} = (1+y)(1+y)^k \geq (1+y)(1+ky) = \]
1. Suppose that \( \lim_{x \to 4} f(x) = L \). Then for any \( \epsilon > 0 \), there exists \( \delta > 0 \) such that if \( 0 < |x - 4| < \delta \), then \( |f(x) - L| < \epsilon \).

3. Show that each of the following sequences is divergent.

(a) \( a_n = 2n \).

**Proof:** Suppose \( (a_n) \) converges. Then by Theorem 48, \( (a_n) \) is bounded. Let \( M \in \mathbb{R} \) be a bound for \( (a_n) \). Then there exists \( n \in \mathbb{N} \) such that \( n > M \), by the Archimedean Property. Thus \( 2n > n > M \), but \( 2n = a_n < M \), a contradiction. Therefore \( (a_n) \) diverges.

(b) \( b_n = (-1)^n \).

**Proof:** Suppose \( (b_n) \to s \in \mathbb{R} \). Let \( \epsilon = \frac{1}{4} \). Then there exists \( N \) such that for all \( n > N \), 
\[
|(-1)^n - s| < \frac{1}{4}.
\]
But if \( n \) is odd we have \(|(-1)^n - s| = |1 - s| = |(-1)(1 + s)| = |1 + s| < \frac{1}{4}.
\]
This implies \( \frac{1}{4} < s < \frac{3}{4} \). On the other hand, if \( n \) is even, \(|(-1)^n - s| = |1 - s| < \frac{1}{4} \). This implies that \( \frac{3}{4} < s < \frac{1}{4} \). But there is no value \( s \in \mathbb{R} \) which satisfies both of those inequalities. Therefore \( (b_n) \) diverges.

(c) \( d_n = (-n)^2 \).

**Proof:** Suppose \( (d_n) \to s \in \mathbb{R} \). Then by Theorem 48, \( \exists M \in \mathbb{R} \) such that \( (d_n) \leq M \) for all \( n \in \mathbb{N} \). By the Archimedian property, there exists \( n \in \mathbb{N} \) such that \( n > M \). But \( d_n = (-n)^2 = n^2 \). Since \( n > M \), \( d_n > M \), a contradiction. Therefore \( (d_n) \) diverges.

5. Suppose that \( \lim s_n = 0 \). If \( (t_n) \) is a bounded sequence, prove that \( \lim (s_n t_n) = 0 \).

**Proof:** Let \( \epsilon > 0 \). Since \( (t_n) \) is bounded, there exists \( m \in \mathbb{R} \) such that \( |t_n| < m \) for all \( n \in \mathbb{N} \). Also since \( \lim s_n = 0 \), there is some \( N_1 \) such that for all \( n > N_1 \), \( |s_n| < \frac{\epsilon}{m} \). Let \( N = \max\{N_1, m\} \). Then for all \( n > N \), we have \( |s_n t_n| = |s_n||t_n| < \frac{\epsilon}{m} (m) = \epsilon \). Therefore \( \lim s_n = 0 \).

6. Prove or give a counterexample: If \( (s_n) \) converges to \( s \), then \( |s_n| \) converges to \( |s| \).

**Proof:** First we will prove a lemma:

**Lemma 1**

\[ |x| - |y| \leq |x - y| \]

for all \( x, y \in \mathbb{R} \).

**Proof:** Suppose \( x, y \geq 0 \). Then \( |x| - |y| = \max(x, y) - \min(x, y) = |x + y| - |x - y| = |x - y| \). If \( x \geq 0 \) and \( y < 0 \), then \( x - y > 0 \), so we have \( |x| - |y| = x + y \leq |x + y| = |x - y| \). If \( x < 0 \) and \( y \geq 0 \), then \( x - y < 0 \), so we have \( |x| - |y| = -x - y = \max(-x, -y) - \min(-x, -y) = (1)(x + y) = x + y \leq |x + y| = |x - y| = -x + y = (-1)(x - y) = |x - y| \). Finally, if \( x, y < 0 \), then \( |x| - |y| = -x + y = \max(-x, -y) - \min(-x, -y) = (-1)(x + y) = |x + y| = |x - y| \). We have covered all of the cases, so we can conclude that \( |x| - |y| \leq |x - y| \) for all \( x, y \in \mathbb{R} \).

Now for the proof of the problem. Let \( \epsilon > 0 \). Then there exists \( N \) such that for all \( n > N \), \( |s_n - s| < \epsilon \). But by the Lemma, we have \( ||s_n| - |s|| \leq |s_n - s| < \epsilon \). Therefore \( (|s_n|) \) converges to \( |s| \).
7. Suppose that \((a_n), (b_n),\) and \((c_n)\) are sequences such that \(a_n \leq b_n \leq c_n\) for all \(n \in \mathbb{N}\) and such that \(\lim a_n = \lim c_n = b\). Prove that \(\lim b_n = b\). (This is called the squeeze theorem.)

Proof: Let \(\epsilon > 0\). Then there exists \(N_1, N_2 \in \mathbb{R}\) such that if \(N = \max\{N_1, N_2\}\), then for all \(n > N\), \(|a_n - b| < \epsilon\) and \(|c_n - b| < \epsilon\). This implies that \(b - \epsilon < a_n \leq b_n \leq c_n < b + \epsilon\). This implies that \(|b_n - b| < \epsilon\). Hence \(\lim b_n = b\).