Math 3210-3
HW 12
Solutions

NOTE: Only turn in problems 1(d), 2(a), 2(c), 3(c), 4(a), and 7.

Sequences

1. Write out the first seven terms of each sequence.

   (a) $a_n = n^2$
   $1, 4, 9, 16, 25, 36, 49, \ldots$

   (b) $b_n = \dfrac{(-1)^n}{n}$
   $-1, \frac{1}{2}, \frac{-1}{3}, \frac{1}{4}, \frac{-1}{5}, \frac{1}{6}, \frac{-1}{7}, \ldots$

   (c) $c_n = \cos \frac{n\pi}{3}$
   $\frac{1}{2}, \frac{-1}{2}, -1, \frac{-1}{2}, \frac{1}{2}, 1, \frac{1}{2}, \ldots$

   (d) $d_n = \dfrac{2n+1}{3n-1}$
   $\frac{3}{2}, 1, \frac{7}{8}, \frac{9}{11}, \frac{11}{14}, \frac{13}{17}, \frac{3}{4} \ldots$

2. Using only the definition of a limit of a sequence, prove the following.

   (a) For any real number k, $\displaystyle\lim_{n\to\infty} \left(\frac{k}{n}\right) = 0$.

   Proof: Let $\epsilon > 0$ and let $N = \dfrac{|k|}{\epsilon}$. Then for all $n > N$, $\left|\dfrac{k}{n} - 0\right| = \left|\dfrac{k}{n}\right| = \dfrac{|k|}{n} < \dfrac{|k|}{N} = \epsilon$. Therefore $\displaystyle\lim_{n\to\infty} \left(\frac{k}{n}\right) = 0$.

   □

   (b) ♣ For any real number $k > 0$, $\displaystyle\lim_{n\to\infty} \left(\frac{1}{n^k}\right) = 0$.

   Proof: Let $\epsilon > 0$ and let $N = \dfrac{1}{\epsilon^{1/k}}$. Then for all $n > N$, we have $\left|\dfrac{1}{n^k} - 0\right| = \dfrac{1}{n^k} < \dfrac{1}{N^k} = \epsilon$.

   □

   (c) $\displaystyle\lim_{n\to\infty} \frac{3n+1}{n+2} = 3$.

   Proof: Let $\epsilon > 0$ and $N = \frac{5}{\epsilon}$. Then for all $n > N$, $\left|\dfrac{3n+1}{n+2} - 3\right| = \left|\dfrac{-5}{n+2}\right| = \dfrac{5}{n+2} < \dfrac{5}{N} = \epsilon$. Therefore $\displaystyle\lim_{n\to\infty} \frac{3n+1}{n+2} = 3$.

   □

   (d) ♣ $\displaystyle\lim_{n\to\infty} \frac{\sin n}{n} = 0$.

   Proof: Let $\epsilon > 0$ and let $N = \frac{1}{\epsilon}$. Then for all $n > N$, $\left|\dfrac{\sin n}{n} - 0\right| = \left|\dfrac{\sin n}{n}\right| \leq \dfrac{1}{n} < \dfrac{1}{N} = \epsilon$. Therefore $\displaystyle\lim_{n\to\infty} \frac{\sin n}{n} = 0$.

   □

(e) $\lim_{n\to\infty} \dfrac{n+2}{n^2-3} = 0.$

Proof: Let $\epsilon > 0$ and $N = \max\{3, \frac{4}{\epsilon}\}$. Then for all $n > N$, $\left|\dfrac{n+2}{n^2-3} - 0\right| = \dfrac{n+2}{n^2-3} < \dfrac{2n}{\frac{1}{2}n^2} = \dfrac{4}{n} \leq \epsilon.$

Therefore $\lim_{n\to\infty} \dfrac{n+2}{n^2-3} = 0.$

□

3. Using any of the Theorems 47-49 or the examples we worked in class from section 4.1, prove the following.

(a) ♣ $\lim_{n\to\infty} \dfrac{1}{1+3n} = 0.$

Proof: Notice that $\left|\dfrac{1}{1+3n} - 0\right| = \dfrac{1}{1+3n} < \dfrac{1}{n}$. And $\lim \dfrac{1}{n} = 0$, so by Theorem 47, $\lim_{n\to\infty} \dfrac{1}{1+3n} = 0.$

□

(b) $\lim_{n\to\infty} \dfrac{4n^2-7}{2n^3-5} = 0$

Proof: If $n \geq 2$, we have $|4n^2 - 7| = 4n^2 - 7 < 4n^2$, and $|2n^3 - 5| = 2n^3 - 5 > n^3$. Therefore $|\dfrac{4n^2-7}{2n^3-5}| < \dfrac{4n^2}{n^3} = 4\left(\dfrac{1}{n}\right)$. Since $\lim \frac{1}{n} = 0$, by Theorem 47, we can conclude that $\lim_{n\to\infty} \frac{4n^2-7}{2n^3-5} = 0.$

□

(c) $\lim_{n\to\infty} \dfrac{6n^2+5}{2n^2-3n} = 3.$

Proof: If $n > 5$, we have $\left|\dfrac{6n^2+5}{2n^2-3n} - 3\right| = \left|\dfrac{9n+5}{2n^2-3n}\right| < \dfrac{10n}{n^2} = \dfrac{10}{n} = 10\dfrac{1}{n}$. Also we proved in class that $\lim \frac{1}{n} = 0$, so by Theorem 47, $\lim_{n\to\infty} \dfrac{3n+1}{n+2} = 3.$

□

(d) ♣ $\lim_{n\to\infty} \dfrac{\sqrt{n}}{n+1} = 0.$

Proof: Notice that $\left|\dfrac{\sqrt{n}}{n+1} - 0\right| = \dfrac{\sqrt{n}}{n+1} < \dfrac{\sqrt{n}}{n} = \dfrac{1}{\sqrt{n}}$. We proved in class that $\lim \frac{1}{\sqrt{n}} = 0$, so once again by Theorem 47, we have $\lim_{n\to\infty} \dfrac{\sqrt{n}}{n+1} = 0.$

□

(e) $\lim_{n\to\infty} \dfrac{n^2}{n!} = 0.$

Proof: I claim that $n^3 < n!$ for $n \geq 6$. Let $n = 6$. Then $6^3 = 216 < 720 = 6!$. Hence the base case is true. Assume that $k^3 < k!$ for some $k \geq 6$. Then $(k+1)^3 = k^3+3k^2+3k+1 < k!+3k^2+3k+1 < k! + k! + k! + \cdots + k!$ where there are $k+1$ summands. Then $(k+1)^3 < (k+1)k! = (k+1)!$. Therefore $n^3 < n!$ for all $n \geq 6$.

So we now have $\left|\dfrac{n^2}{n!}\right| = \dfrac{n^2}{n!} < \dfrac{n^2}{n^3} = \dfrac{1}{n}$. We proved in class that $\lim \frac{1}{n} = 0$, so by Theorem 47, $\lim_{n\to\infty} \dfrac{n^2}{n!} = 0.$

□

(f) If $|x| < 1$, then $\lim_{n\to\infty} x^n = 0$.

Proof: First, if $x = 0$, then clearly $\lim_{n\to\infty} x^n = 0$. So suppose $x \neq 0$. Then $|x| < 1$, so there is some $y \in \mathbb{R}$ such that $|x| = \frac{1}{1+y}$. I claim that $(1+y)^n \geq 1 + ny$ for all $n \in \mathbb{N}$. I will prove this claim by induction. If $n = 1$, $(1+y) = 1 + 1y$, so the claim is true for $n = 1$. Now suppose $(1+y)^k \geq 1 + ky$ for some $k \in \mathbb{N}$. Then $(1+y)^{k+1} = (1+y)(1+y)^k \geq (1+y)(1+ky) =$

$1+ky+y+ky^2 = 1+y(k+1)+ky^2 > 1+y(k+1)$, so $n = k+1$ is true. Therefore $(1+y)^n \geq 1+ny$ for all $n \in \mathbb{N}$.

Hence we have $|x|^n = \left(\frac{1}{1+y}\right)^n \leq \frac{1}{1+ny} < \frac{1}{ny} = \frac{1}{y}\left(\frac{1}{n}\right)$. Since $\frac{1}{n} \to 0$, then by Theorem 47, $\lim_{n\to\infty} |x|^n = 0$.

□

4. Show that each of the following sequences is divergent.

(a) $a_n = 2n$.

Proof: Suppose (a_n) converges. Then by Theorem 48, (a_n) is bounded. Let $M \in \mathbb{R}$ be a bound for (a_n). Then there exists $n \in \mathbb{N}$ such that $n > M$, by the Archimedean Property. Thus $2n > n > M$, but $2n = a_n < M$, a contradiction. Therefore (a_n) diverges.

□

(b) ♣ $b_n = (-1)^n$.

Proof: Suppose $(b_n) \to s \in \mathbb{R}$. Let $\epsilon = \frac{1}{4}$. Then there exists N such that for all $n > N$, $|(-1)^n - s| < \frac{1}{4}$. But if n is odd we have $|(-1)^n - s| = |-1-s| = |(-1)(1+s)| = |1+s| < \frac{1}{4}$. This implies $\frac{-5}{4} < s < \frac{-3}{4}$. On the other hand, if n is even, $|(-1)^n - s| = |1-s| < \frac{1}{4}$. This implies that $\frac{3}{4} < s < \frac{5}{4}$. But there is no value $s \in \mathbb{R}$ which satisfies both of those inequlities. Therefore (b_n) diverges.

□

(c) $d_n = (-n)^2$.

Proof: Suppose $(d_n) \to s \in \mathbb{R}$. Then by Theorem 48, $\exists M \in \mathbb{R}$ such that $(d_n) \leq M$ for all $n \in \mathbb{N}$. By the Archimedean property, there exists $n \in \mathbb{N}$ such that $n > M$. But $d_n = (-n)^2 = n^2 > n > M$, a contradiction. Therefore (d_n) diverges.

□

5. Suppose that $\lim s_n = 0$. If (t_n) is a bounded sequence, prove that $\lim(s_n t_n) = 0$.

Proof: Let $\epsilon > 0$. Since (t_n) is bounded, there exists $m \in \mathbb{R}$ such that $|t_n| < m$ for all $n \in \mathbb{N}$. Also since $\lim s_n = 0$, there is some N_1 such that for all $n > N_1$, $|s_n| < \frac{\epsilon}{m}$. Let $N = \max\{N_1, m\}$. Then for all $n > N$, we have $|s_n t_n - 0| = |s_n t_n| = |s_n||t_n| < \frac{\epsilon}{m}(m) = \epsilon$. therefore $\lim s_n = 0$.

□

6. Prove or give a counterexample: If (s_n) converges to s, then $(|s_n|)$ converges to $|s|$.

Proof: First we will prove a lemma:

Lemma 1
$||x| - |y|| \leq |x-y|$ for all $x, y \in \mathbb{R}$.

Proof: Suppose $x, y \geq 0$. Then $||x|-|y|| = |x-y|$. If $x \geq 0$ and $y < 0$, then $x - y > 0$, so we have $||x|-|y|| = |x-(-y)| = |x+y| \leq |x|+|y| = x-y = |x-y|$. If $x < 0$ and $y \geq 0$, $x-y < 0$, so we have $||x|-|y|| = |-x-y| = |(-1)(x+y)| = |x+y| \leq |x|+|y| = -x+y = (-1)(x-y) = |x-y|$. Finally, if $x, y < 0$, then $||x|-|y|| = |-x+y| = |(-1)(x-y)| = |x-y|$. We have covered all of the cases, so we can conclude that $||x|-|y|| \leq |x-y|$ for all $x, y \in \mathbb{R}$.

□

Now for the proof of the problem. Let $\epsilon > 0$. Then there exists N such that for all $n > N$, $|s_n - s| < \epsilon$. But by the Lemma, we have $||s_n| - |s|| \leq |s_n - s| < \epsilon$. Therefore $(|s_n|)$ converges to $|s|$.

□

7. Suppose that $(a_n), (b_n)$, and (c_n) are sequences such that $a_n \leq b_n \leq c_n$ for all $n \in \mathbb{N}$ and such that $\lim a_n = \lim c_n = b$. Prove that $\lim b_n = b$. (This is called the squeeze theorem.)

   Proof: Let $\epsilon > 0$. Then there exists $N_1, N_2 \in \mathbb{R}$ such that if $N = \max\{N_1, N_2\}$, then for all $n > N$, $|a_n - b| < \epsilon$ and $|c_n - b| < \epsilon$. This implies that $b - \epsilon < a_n \leq b_n \leq c_n < b + \epsilon$. This implies that $|b_n - b| < \epsilon$. Hence $\lim b_n = b$.

   □