§3.2: Ordered Fields

$\mathbb{R}$ is a complete ordered field. We will explain what this means.

can think $R = \mathbb{R}$

Assume we have a set R with two operations, addition $+$ and multiplication $\cdot$ satisfying:

A1: $\forall x, y \in R$, $x+y \in R$. If $x = w$ and $y = z$ then $x+y = w+z$

A2: $\forall x, y \in R$, $x+y = y+x$ (commutative law of addition)

A3: $\forall x, y, z \in R$, $x+(y+z) = (x+y)+z$ (associative law of addition)

A4: $\exists!$ number $0 \in R$ s.t. $x + 0 = x$ $\forall x \in R$.

A5: For each $x \in R$ $\exists!$ number $y \in R$ s.t. $x+y = 0$. i.e. $y = -x$.

M1: $\forall x, y \in R$, $x \cdot y \in R$ and if $x = w$ ~~and~~ and $y = z$, then $x \cdot y = w \cdot$

M2: $\forall x, y \in R$, $x \cdot y = y \cdot x$ (Commutative law of multiplication)

M3: $\forall x, y, z \in R$, $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ (associative law of multiplication)

M4: $\exists!$ number $1 \in R$ s.t. $1 \neq 0$ and $x \cdot 1 = x$ $\forall x \in R$.

M5: For each $x \in R$ with $x \neq 0$ $\exists!$ $x^{-1} \in R$ s.t. $x \cdot x^{-1} = 1$.

DL: For any $x, y, z \in R$, $x \cdot (y+z) = x \cdot y + x \cdot z$. (Distributive law)

These are the field axioms. Any system (a set with two operations) which satisfy these axioms is called a <u>field</u>.

We can think of $+$ and $\cdot$ as maps from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$.

Ex: Which axioms does $\mathbb{N}$ satisfy? (Not A4, not A5) (Not M5)

Which axioms does $\mathbb{Z}$ satisfy? (All but M5) ← Commutative Ring

Which axioms does $\mathbb{Q}$ satisfy? All ← field

~~Ex~~ Let M = set of all 2×2 matrices. Which axioms does M satisfy? (Not M2 or M5) ← Noncommutative ring.

The real numbers also satisfy the four order axioms:

O1: For $x, y \in R$, exactly one of the relations $x=y$, $x<y$, or $y<x$ holds. (Trichotomy law)

O2: $\forall x,y,z \in R$, if $x<y$ and $y<z$ then $x<z$

O3: $\forall x,y,z \in R$, if $x<y$ then $x+z<y+z$

O4: $\forall x,y \in R$, if $x<y$ and $z>0$ then $xz<yz$.

Theorem 25: Let $x,y,z \in \mathbb{R}$

a) If $x+z=y+z$ then $x=y$

b) $x \cdot 0 = 0$

c) $(-1) \cdot x = -x$

d) $xy=0$ iff $x=0$ or $y=0$

e) $x<y$ iff $-y<-x$

f) If $x<y$ and $z<0$ then $xz>yz$.

Pf: (a) If $x+z=y+z$ then

$(x+z)+(-z)=(y+z)+(-z)$ by A5 and A1

$\Rightarrow x+(z+(-z)) = y+(z+(-z))$ by A3

$\Rightarrow x+0=y+0$ by A5

$\Rightarrow x=y$ by A4. ∎

(b) $\forall x \in \mathbb{R}$ we have

$x \cdot 0 = x(0+0)$ by A4

$\Rightarrow x\cdot 0 = x\cdot 0 + x \cdot 0$ by DL

$\Rightarrow 0 + x\cdot 0 = x \cdot 0 + x \cdot 0$ by A4 and A2

$\Rightarrow 0 = x \cdot 0$ by part (a) ∎

(c) $\forall x \in \mathbb{R}$ we have

$$\begin{aligned}
x + (-1)\cdot x &= x + x(-1) && \text{by M2} \\
&= x\cdot 1 + x(-1) && \text{by M4} \\
&= x\,[1 + (-1)] && \text{by DL} \\
&= x \cdot 0 && \text{by A5} \\
&= 0 && \text{by (b).}
\end{aligned}$$

Thus $(-1)\cdot x = -x$ by the uniqueness of $-x$ in A5. $\boxed{\text{Ⓢ}}$

(d) Exercise

(e) Suppose that $x < y$. Then

$$\begin{aligned}
& x + [(-x) + (-y)] < y + [(-x) + (-y)] && \text{by O3} \\
\Rightarrow\ & x + [(-x) + (-y)] < y + [(-y) + (-x)] && \text{by A2} \\
\Rightarrow\ & [x + (-x)] + (-y) < [y + (-y)] + (-x) && \text{by A3} \\
\Rightarrow\ & 0 + (-y) < 0 + (-x) && \text{by A5} \\
\Rightarrow\ & -y < -x && \text{by A2 and A4.}
\end{aligned}$$

The converse is similar. $\boxed{\text{Ⓢ}}$

(f) Exercise

Any system which satisfies these 15 axioms is an <u>ordered field</u>.

Ex: $\mathbb{R}$, $\mathbb{Q}$.

Ex: Let F be the set of all rational functions, i.e. F is the set of all quotients of polynomials. A typical element in F look like:

$$P = \frac{a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0}{b_m x^m + b_{m-1} x^{m-1} + \dots + b_1 x + b_0}$$

with $a_i, b_i \in \mathbb{R}$ and $b_m \neq 0$.

Using regular addition and multiplication we see F is a field. We can define an order on F: $p > 0$ iff a_n and b_m have the same sign, i.e. $a_n b_m > 0$.

Eg: $\dfrac{3x^2 + 4x - 1}{7x^5 + 5} > 0$ since $3 \cdot 7 > 0$.

If $\frac{p}{q}$ and $\frac{f}{g}$ are rational functions, we say $\frac{p}{q} > \frac{f}{g}$ iff $\frac{p}{q} - \frac{f}{g} > 0$.

It is an exercise to show this satisfies the order axioms.

So F is an ordered field.

Theorem 26: Let $x, y \in \mathbb{R}$ s.t. $x \leq y + \varepsilon$ $\forall \varepsilon > 0$. Then $x \leq y$.

pf: (Contrapositive) By axiom O1, the negation of $x \leq y$ is $x > y$. Thus we suppose $x > y$ and we must show $\exists \varepsilon > 0$ s.t. $x > y + \varepsilon$. Let $\varepsilon = \frac{x-y}{2}$. Since $x > y$, $\varepsilon > 0$. Furthermore

$$y + \varepsilon = y + \frac{x-y}{2} = \frac{x+y}{2} < \frac{x+x}{2} = x$$ as required.

∎

Theorem 27: Let $x, y \in \mathbb{R}$ and let $a \geq 0$. Then

a) $|x| \geq 0$

b) $|x| \leq a$ iff $-a \leq x \leq a$

c) $|x \cdot y| = |x||y|$

d) $|x+y| \leq |x| + |y|$ Triangle inequality

Pf: a) There are two cases: If $x \geq 0$ then $|x| = x \geq 0$. On the other hand if $x < 0$, $|x| = -x > 0$. In both cases $|x| \geq 0$.

b) Since $|x| = x$ or $-|x| = x$ it follows that $-|x| \leq x \leq |x|$. So if $|x| \leq a$, then

$$-a \leq -|x| \leq x \leq |x| \leq a$$

Conversely suppose $-a \leq x \leq a$. If $x \geq 0$ then $|x| = x \leq a$, and if $x < 0$, $|x| = -x \leq a$, so $|x| \leq a$.

c) Exercise

d) As in (b) we have

$-|x| \leq x \leq |x|$ and $-|y| \leq y \leq |y|$. Adding these we get $-(|x| + |y|) \leq x + y \leq |x| + |y|$, so by part (b),

$\Rightarrow |x+y| \leq |x| + |y|$. ∎