Math 3210-3   Exam 2

Solutions

Name:________________________________

You may use your dictionary of definitions. Please use a pencil and keep your proofs neat and organized. Make sure you use complete "sentences," and remember you need to give justifications for each step in your proofs. Each problem is worth one point. Write on this front page which problems you want graded. Notice that you have some choice as to which problems you will do. You may refer to theorems we proved in class by either stating the name of the theorem or giving a brief synopsis of the statement of the theorem.

Grade Problems: ____

Do one of the following:

1. State the Intermediate Value Theorem.

   Suppose that $f : [a, b] \to \mathbb{R}$ is continuous. Then f has the intermediate value property on $[a, b]$. That is, if k is any value between $f(a)$ and $f(b)$ (i.e. $f(a) < k < f(b)$ or $f(b) < k < f(a)$), then there exists $c \in [a, b]$ such that $f(c) = k$.

2. State the Bolzano-Weierstrass Theorem for Sequences.

   Every bounded sequence has a convergent subsequence.

Do one of the following:

3. True of false: If $(|s_n|)$ converges, then (s_n) converges. You will need to prove your assertion.

   This is false. For a counterexample, let $(s_n) = (-1, 1, -1, 1, \ldots)$, so $s_n = (-1)^n$ for each $n \in \mathbb{N}$. Then $(|s_n|) = (1, 1, 1, 1, \ldots)$ which clearly converges to 1. On the other hand, consider the following subsequences of (s_n). Let $(a_n) = (1, 1, 1, 1 \ldots)$ and $(b_n) = (-1, -1, -1, \ldots)$. Then (a_n) converges to 1 and (b_n) converges to -1. We know if a sequence converges to some value s, then all of its subsequences converge to s. Since $1 \neq -1$, (s_n) does not converge.

4. Define a sequence (s_n) to be $s_1 = 4$ and $s_{n+1} = \frac{s_n^2+4}{2s_n}$. Assuming that s_n is monotonic, prove the sequence converges, and find its limit. (The sequence is monotonic, but I don't want you to take the time to show it. You can if you want, but you don't need to.)

   Proof: First notice that $s_1 = 4$, so $s_2 = \frac{4^2+4}{2\cdot 4} = \frac{5}{2}$. Since we are assuming that (s_n) is monotonic, and we see that $s_1 = 4 > \frac{5}{2} = s_2$, we must have that (s_n) is non increasing. Thus it is bounded above by 4. I claim that $s_n > 0$ for each $n \in \mathbb{N}$. We will show this by induction. Clearly $s_1 = 4 > 0$. Now assume $s_k > 0$ for some $k \in \mathbb{N}$. Then $s_{k+1} = \frac{s_k^2+4}{2s_k} > 0$. Thus 0 is a lower bound for (s_n). Since (s_n) is a bounded monotonic sequence, the Monotone Convergence Theorem implies that (s_n) converges.

   To find the value (s_n) converges to, notice that $\lim_{n\to\infty} s_n = \lim_{n\to\infty} s_{n+1}$. Since we know the sequence converges, we can say it converges to $s \in \mathbb{R}$. So s must satisfy $s = \frac{s^2+4}{2s}$. We can solve for s, and we see $s = \pm 2$. Since $s_n > 0$ for each n, we must have $s > 0$, so $s = 2$.

   □

Do one of the following:

5. Suppose that f is a continuous function on $[0, 1]$ and that $f(x)$ is in $[0, 1]$ for each x. Prove that $f(x) = x$ for some $x \in [0, 1]$. In other words, prove f has a fixed point.

   Proof: Let $g(x) = f(x) - x$. Then g is a continuous function since the sum of continuous functions is continuous, and $g(0) = f(0) \geq 0$ since $f(0) \in [0, 1]$. Also $g(1) = f(1) - 1 \leq 0$ since $f(1) \in [0, 1]$. Thus by the intermediate value theorem, there is some $x \in [0, 1]$ such that $g(x) = 0$ which implies for this x, $f(x) = x$.

   □

6. Use the definition of uniformly continuous to prove $f(x) = \frac{1}{x}$ is uniformly continuous on $[\frac{1}{2}, \infty)$.

   Proof: Let $\epsilon > 0$ and let $\delta = \dfrac{\epsilon}{4}$. Then if $x, y \in \left[\dfrac{1}{2}, \infty\right)$ and $|x - y| < \delta$, then we have $|f(x) - f(y)| = \left|\dfrac{1}{x} - \dfrac{1}{y}\right| = \left|\dfrac{y - x}{xy}\right| < \left|\dfrac{\delta}{\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)}\right| = 4\delta = \epsilon$. Therefore f is uniformly continuous on $\left[\dfrac{1}{2}, \infty\right)$.

   □

Prove one of the following:

7. Suppose that (s_n) and (t_n) are convergent sequences with $\lim s_n = s$ and $\lim t_n = t$. Then $\lim s_n t_n = st$.

   Proof: Let $\epsilon > 0$. We proved in class that every convergent sequence is bounded, so since (s_n) converges, there is some $M_1 \in \mathbb{R}$ such that $|s_n| < M_1$ for all $n \in \mathbb{N}$. Let $M = \max\{M_1, |t|\}$. We also have that since (s_n) and (t_n) converge, there is some $N_1, N_2 \in \mathbb{R}$ such that if $n > N_1$ then $|s_n - s| < \frac{\epsilon}{2M}$, and if $n > N_2$ then $|t_n - t| < \frac{\epsilon}{2M}$. Let $N = \max\{N_1, N_2\}$. Then for $n > N$ we have the following:

$$\begin{aligned} |s_n t_n - st| &= |s_n t_n - s_n t + s_n t - st| \\ &= |s_n(t_n - t) + t(s_n - s)| \\ &\leq |s_n||t_n - t| + |t||s_n - s| \\ &< M\frac{\epsilon}{2M} + M\frac{\epsilon}{2M} \\ &= \epsilon \end{aligned}$$

   Therefore $\lim s_n t_n = st$.

   □

8. Let $f : D \to \mathbb{R}$ and $g : E \to \mathbb{R}$ be functions such that $f(D) \subseteq E$. If f is continuous at a point $c \in D$ and g is continuous at $f(c)$, then the composition $g \circ f : D \to \mathbb{R}$ is continuous at c.

   Proof: Let $e = f(c)$, and let W be a neighborhood of $g(e)$. Since g is continuous at e, there exists a neighborhood V of e such that $g(V \cap E) \subseteq W$. Since $e = f(c)$ and f is continuous at c, there exists a neighborhood U of c such that $f(U \cap D) \subseteq V$. Since $f(D) \subseteq E$ we have $f(U \cap D) \subseteq (V \cap E)$, so $g(f(U \cap D)) \subseteq W$. Therefore by Theorem 69 $g \circ f$ is continuous at c.

   □