CH. 8: Infinite Series

§8.1: Convergence of Infinite Series

Let (a_n) be a sequence of real numbers. We use the notation $\sum_{k=m}^{n} a_k$ to denote the sum $a_m + a_{m+1} + \dots + a_n$ where $n \geq m$. Using (a_n) we can define a new sequence (s_n) of partial sums given by

$$s_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \dots + a_n.$$

We also refer to the sequence (s_n) of partial sums as the infinite series or simply the series

$$\sum_{n=1}^{\infty} a_n.$$

If (s_n) converges to a real number s, we say that the series $\sum_{n=1}^{\infty} a_n$ is convergent and we write $\sum_{n=1}^{\infty} a_n = s$.

We also refer to s as the sum of the series $\sum_{n=1}^{\infty} a_n$.

A series that doesn't converge is divergent. If $\lim_{n\to\infty} s_n = +\infty$ we say the series $\sum_{n=1}^{\infty} a_n$ diverges to $+\infty$ and write $\sum_{n=1}^{\infty} a_n = +\infty$.

Note: $\sum_{n=1}^{\infty} a_n$ denotes two different things

$$\sum_{n=1}^{\infty} a_n = (s_n) = \text{sequence of partial sums}$$

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n$$

Context will make it clear which is meant.

We don't have to start @ $n=1$.

Ex: Consider the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

The partial sums are

$$S_n = \sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \dots + \frac{1}{n(n+1)}$$

$$= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \dots + \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$= 1 - \frac{1}{n+1}$$

This is a "telescoping" series.

Also $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right) = 1.$

Ex: Harmonic Series: $\sum_{n=1}^{\infty} \frac{1}{n}$

has partial sums $S_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$

In a previous example we showed (s_n) diverges, thus $\sum \frac{1}{n}$ diverges.

Since (s_n) is an increasing sequence $\sum \frac{1}{n} = +\infty$.

Theorem 104: Suppose $\sum a_n = s$ and $\sum b_n = t$. Then $\sum (a_n + b_n) = s+t$ and $\sum (k a_n) = ks$ $\forall k \in \mathbb{R}$.

Pf: Follows from the similar Theorem involving sequences $\blacksquare$

Theorem 105: If $\sum a_n$ is a convergent series, then $\lim a_n = 0$.

Pf: If $\sum a_n$ converges, the sequence of partial sums (s_n) must have a finite limit. But $a_n = s_n - s_{n-1}$

so $\lim a_n = \lim s_n - \lim s_{n-1} = 0$ since $\lim s_n = s = \lim s_{n-1}$. $\Rightarrow$

Theorem 106: (Cauchy Criterion for Series) The infinite series $\sum a_n$ converges iff $\forall \varepsilon > 0$ $\exists N \in \mathbb{R}$ s.t. if $n \geq m > N$, then $|a_m + a_{m+1} + \dots + a_n| < \varepsilon$.

Pf: Suppose that $\sum a_n$ converges. Then the sequence of partial sums (s_n) converges, so by ~~Theorem~~ Lemma 2), (s_n) is Cauchy. Thus given any $\varepsilon > 0$ $\exists N$ s.t. $m, n > N \Rightarrow |s_n - s_m| < \varepsilon$. Hence if $n \geq m > N+1$, then $m-1 > N$, so that

$$|a_m + a_{m+1} + \dots + a_n| = |s_n - s_{m-1}| < \varepsilon.$$

Conversely, given any $\varepsilon > 0$, suppose $\exists N$ s.t. $n \geq m > N \Rightarrow |a_m + a_{m+1} + \dots + a_n| < \varepsilon$. Then for $n > m > N$ we have $m+1 > N$ so that $|s_n - s_m| = |a_{m+1} + a_{m+2} + \dots + a_n| < \varepsilon$.

Thus the sequence (s_n) of partial sums is Cauchy + the series is convergent. $\square$

Ex: One of the most useful series is the <u>geometric series</u> $\sum_{n=0}^{\infty} r^n$. In a HW problem we saw for $r \neq 1$

$$s_n = \sum_{k=0}^{n} r^k = 1 + r + r^2 + \dots + r^n = \frac{1-r^{n+1}}{1-r} \quad \forall n \in \mathbb{N}$$

$\sum_{n=0}^{\infty} r^n = \lim s_n = \frac{1}{1-r}$

If $|r| \geq 1$ (r^n) does not converge to 0, so by Theorem (105), $\sum r^n$ diverges.

Note: If $\sum a_n$ converges then $\lim a_n = 0$ True

If $\lim a_n \neq 0$, then $\sum a_n$ diverges. True

If $\lim a_n = 0$, then $\sum a_n$ converges. False

Ex: $\sum \frac{1}{n}$.