§5.2: Continuous Functions

$\lim_{x \to c} f(x)$ does not care what $f(c)$ is or even if it exists. Now that will matter to us!

Def: Let $f: D \to \mathbb{R}$ and let $c \in D$. f is <u>continuous</u> at c if
$\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. $|f(x) - f(c)| < \varepsilon$ whenever $|x - c| < \delta$, and $x \in D$.

If f is continuous at each point of a subset S of D, then f is <u>continuous on S</u>. If f is continuous on its domain, then f is <u>continuous</u>.

Note: Continuous @ $c \Rightarrow c \in D$, but c need not be an accumulation point of D. In fact, suppose c is an isolated point of D. And let $\varepsilon > 0$. Then $\exists N(c; \delta)$ s.t. $N \cap D = \{x\} \Rightarrow$ whenever $|x - c| < \delta \Rightarrow x = c \Rightarrow |f(x) - f(c)| = 0 < \varepsilon$. $\Rightarrow f$ continuous @ c. So continuity is slightly more general then limit.

Theorem 69: Let $f: D \to \mathbb{R}$ and let $c \in D$. TFAE:

a) f is continuous at c.

b) If (x_n) is any sequence in D s.t. $(x_n) \to c$, then $\lim f(x_n) = f(c$

c) For every Nbhd V of $f(c)$ there exists a Nbhd U of c s.t $f(U \cap D) \subseteq V$.

Furthermore, if c is an accumulation point of D, then the abov are equivalent to

d) f has a limit at c and $\lim_{x \to c} f(x) = f(c)$.

Pf: Suppose first that c is an isolated point of D. Then $\exists$ Nbhd U of c s.t. $U \cap D = \{c\}$. So for any Nbhd V of $f(c)$, $f(U \cap D) = f(\{c\}) \subseteq$ Thus c always holds. Similarly by an exercise, if (x_n) is a sequence D converging to c, then $x_n \in U$ for all $n > M$, for some M. This

implies $x_n = c$ for $n > M \Rightarrow \lim f(x_n) = f(c)$, so (b) also holds.
We have already shown (a) holds, so (a), (b), and (c) are all equivalent.

Now suppose c is an accumulation point of D. Then (a) $\Leftrightarrow$ (d) is by definition of continuous, (d) $\Leftrightarrow$ (c) by Theorem (20.2) and (d) $\Leftrightarrow$ (b) is essentially Theorem (20.8). ∎

Ex: Let $p(x)$ be a polynomial. Then we saw $\lim_{x \to c} p(x) = p(c)$, so p is continuous at c. $\Rightarrow$ Any polynomial function is continuous on $\mathbb{R}$.

Ex: Let $f(x) = x \sin\left(\frac{1}{x}\right)$, for $x \neq 0$ and $f(0) = 0$.
We will prove $f(x)$ is continuous at 0.

$$|f(x) - f(0)| = |f(x)| = \left|x \sin\left(\tfrac{1}{x}\right)\right| \leq |x| \quad \forall x.$$

Given $\varepsilon > 0$, let $\delta = \varepsilon$. Then when $|x - 0| = |x| < \delta = \epsilon$, we have $|f(x) - 0| = |f(x)| = \left|x \sin\left(\frac{1}{x}\right)\right| \leq |x| < \varepsilon$.
$\therefore$ f is continuous at 0. ∎

We shall later prove f is continuous on $\mathbb{R}$.

Theorem 70: Let $f : D \to \mathbb{R}$ and let $c \in D$. Then f is discontinuous at c iff there exists a sequence (x_n) in D s.t. (x_n) converges to c, but $(f(x_n))$ does not converge to $f(c)$.

Pf: Simply the negation of (a) and (d) in the previous Theorem. ∎

Ex: Let $f(x) = \frac{1}{x}$ $\forall x \in D = (-\infty, 0) \cup (0, \infty)$. Since $\lim_{x \to c} \frac{1}{x} = \frac{1}{c}$ for all $c \in D$, f is continuous on D. But f is not continuous on $\mathbb{R}$. First of all $0 \notin D \Rightarrow f(0)$ is not defined, so it can not be continuous at 0. Secondly, if we were to define $f(0) = k$ for some $k \in \mathbb{R}$, f would still be discontinuous at 0. Indeed, since $\frac{1}{n} \to 0$ and $\lim f\left(\frac{1}{n}\right) = +\infty$, the sequence $\left(f\left(\frac{1}{n}\right)\right)$ diverges.

Ex: Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

If $c \in \mathbb{R}$, every Nbhd of c contains rational points where $f(x) = 1$ and irrational points where $f(x) = 0$. Thus $\lim_{x \to c} f(x)$ can not exist. $\therefore$ f is discontinuous at every point in $\mathbb{R}$.

Ex: Let $f : (0,1) \to \mathbb{R}$ be the Dirichlet function given by

$$f(x) = \begin{cases} \frac{1}{n} & \text{if } x = \frac{m}{n} \text{ is rational in lowest terms} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

So $f(\frac{1}{3}) = f(\frac{2}{3}) = \frac{1}{3}$, $f(\frac{3}{5}) = \frac{1}{5}$, $f(\frac{\sqrt{2}}{5}) = 0, \dots$

We claim f is continuous at each irrational number in $(0,1)$ and discontinuous at each rational number in $(0,1)$.

Pf: Suppose $x \in (0,1)$ and x is rational. Let (x_n) be a sequence of irrational numbers in $(0,1)$ which converges to c. Then $f(x_n) = 0$ $\forall n$, So $\lim f(x_n) = 0 \neq f(c)$ since $f(c) > 0$. $\therefore$ f is discontinuous at c.

On the other hand, let $d \in (0,1)$ and d irrational. Given any $\varepsilon > 0$, by the Archimedean Property, $\exists K \in \mathbb{N}$ s.t. $\frac{1}{K} < \varepsilon$. There are only a finite number of rationals in $(0,1)$ whose denominators are less then K. Thus $\exists \delta > 0$ s.t. all the rationals in $(d - \delta, d + \delta)$ have a denominator (in lowest terms) greater then or equal to K. $\therefore$ If $x \in (0,1)$ and $|x - d| < \delta$, $|f(x) - f(d)| = |f(x)| \leq \frac{1}{K} < \varepsilon$. $\therefore$ f is continuous at d.

Theorem 71: Let f and g be functions from D to $\mathbb{R}$, and let $c \in D$. If f and g are continuous at c, then

a) $f+g$ and fg are continuous at c, and

b) $\frac{f}{g}$ is continuous at c if $g(c) \neq 0$.

Pf: We have a similar Theorem for sequences, and since continuity can be expressed in terms of limits of sequences, we can do the following:

Let (x_n) be a sequence in D converging to c. We need only show $\lim (f+g)(x_n) = (f+g)(c)$. So we have

$$\begin{aligned} \lim (f+g)(x_n) &= \lim \left(f(x_n) + g(x_n)\right) && \text{by def'n.} \\ &= \lim f(x_n) + \lim g(x_n) && \text{by Theorem 53} \\ &= f(c) + g(c) = (f+g)(c). \end{aligned}$$

The others are similar. But for $g(x_n)$ choose (x_n) so that $g(x_n) \neq 0 \ \forall n$. ∎

Theorem 72: Let $f: D \to \mathbb{R}$ and $g: E \to \mathbb{R}$ s.t. $f(D) \subseteq E$. If f is continuous at $c \in D$ and g is continuous at $f(c) \in E$, then $g \circ f: D \to \mathbb{R}$ is continuous at c.

Pf: Let $e = f(c)$, and let W be a nbhd of $g(e)$. Since g is continuous at e, $\exists V$ a nbhd of e s.t. $g(V \cap E) \subseteq W$. Since $e = f(c)$ and f is continuous at c, $\exists U$ a nbhd of c s.t. $f(U \cap D) \subseteq V$. Since $f(D) \subseteq E$, we have $f(U \cap D) \subseteq (V \cap E)$, so $g(f(U \cap D)) \subseteq W$. $\therefore$ By Theorem 69 $g \circ f$ is continuous at c. ∎

skip

Ex: Let $g(x) = \sin x$, $h(x) = \frac{1}{x}$ and $i(x) = x$. Then $\forall x \neq 0$ we have

$$x \sin\left(\tfrac{1}{x}\right) = \left[(i)(g \circ h)\right](x)$$

If we assume $\sin x$ is continuous for all x, then by our Theorems, $x \sin\left(\frac{1}{x}\right)$ is continuous for all $x \neq 0$. By our previous examp[le] if $f(x) = x \sin\left(\frac{1}{x}\right)$ for $x \neq 0$ and $f(0) = 0$, f is continuous on $\mathbb{R}$