§3.5: Compact Sets

Def: If $\mathcal{F}$ is a collection of open sets whose union contains a set S, then $\mathcal{F}$ is an open cover of S. If $\mathcal{G} \subseteq \mathcal{F}$ and $\mathcal{G}$ is also an open cover of S, the $\mathcal{G}$ is called a subcover of S.

A set S is compact if every open cover of S contains a finite subcover.

Ex: a) Let $S = (0,2)$, and for each $n \in \mathbb{N}$ let $A_n = (\frac{1}{n}, 3)$. If $0 < x < 2$ then by the Archimedean property (Thrm 37(c)), $\exists p \in \mathbb{N}$ s.t. $\frac{1}{p} < x$. Thus $x \in A_p$ and $\mathcal{F} = \{A_n : n \in \mathbb{N}\}$ is an open cover of S. However if $\mathcal{G} = \{A_{n_1}, A_{n_2}, \ldots, A_{n_k}\}$ is any finite subfamily of $\mathcal{F}$, and if $m = \max\{n_1, \ldots, n_k\}$, then

$$A_{n_1} \cup \cdots \cup A_{n_k} = A_m = \left(\frac{1}{m}, 3\right)$$

So a finite subfamily $\mathcal{G}$ is not an open cover of $(0,2)$

$\therefore$ $(0,2)$ is not compact.

b) Let $S = \{x_1, \ldots, x_n\}$ be a finite subset of $\mathbb{R}$, and let $\mathcal{F} = \{A_\alpha\}$ be an open cover of S. Then for each $i = 1, \ldots, n$ $\exists A_{\alpha_i} \in \mathcal{F}$ s.t. $x_i \in A_{\alpha_i}$ $\Rightarrow$ $A_{\alpha_1} \cup A_{\alpha_2} \cup \cdots \cup A_{\alpha_n}$ is a finite open cover of S.

$\therefore$ any finite set is compact.

Exercise: Show $[0, \infty)$ is not compact.

Note: To show S is compact $\Rightarrow$ show every open cover has a finite subcover.

To show S is not compact $\Rightarrow$ construct an open cover

Lemma 1: If S is a nonempty closed bounded subset of $\mathbb{R}$, then S has a maximum and a minimum.

Pf: Since S is bounded above $m = \sup S$ exists by the completeness axiom. If $m \notin S$ then for each $\varepsilon > 0$ $\exists x$ in S s.t. $m - \varepsilon < x < m + \varepsilon \Rightarrow m$ is an accumulation point for S. But S is closed $\Rightarrow m \in S$. $\therefore m = \max S$. Similarly $\inf S \in S$, so $\inf S = \min S$. $\blacksquare$

Theorem 44: (Heine-Borel) A subset S of $\mathbb{R}$ is compact iff S is closed and bounded.

Pf: Suppose S is compact. For each $n \in \mathbb{N}$, let $I_n = N(0; n) = (-n,$ Then each I_n is open and $S \subseteq \bigcup_{n=1}^{\infty} I_n$. Thus $\{I_n\}$ is an open cover of S. Since S is compact, $\exists$ finitely many integers $n_1, \ldots, n_k$ s.t. $S \subseteq (I_{n_1} \cup \cdots \cup I_{n_k}) = I_m$, where $m = \max\{n_1, \ldots, n_k\}$. So $\forall x \in S$, $|x| < m$, and S is bounded.

Now suppose S is not closed. Then $\exists p \in (\mathrm{cl}\, S) \setminus S$. For each $n \in \mathbb{N}$, we let $U_n = \mathbb{R} \setminus \mathrm{cl}\, N(p; \frac{1}{n})$. Each U_n is open and we have

$$\bigcup_{n=1}^{\infty} U_n = \mathbb{R} \setminus \bigcap_{n=1}^{\infty} \mathrm{cl}\, N(p; 1/n) = \mathbb{R} \setminus \{p\} \supseteq S,$$ by a HW problem

Thus $\{U_n\}$ is an open cover of S. S is compact, so $\exists n_1, \ldots, n_k \in \mathbb{N}$ s.t. $S \subseteq \{U_{n_1} \cup \cdots \cup U_{n_k}\}$. But the U_n's are nested. Thus $U_m \subseteq U_n$ if $m \leq n$. So $S \subseteq U_{n_k}$. But $S \cap N(p; 1/n_k) = \emptyset$, contradicting our choice of $p \in (\mathrm{cl}\, S) \setminus S$.

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Conversely, suppose S is closed and bounded. Let $\mathcal{F}$ be an open cover of S. For each $x \in \mathbb{R}$ let $S_x = S \cap (-\infty, x]$ and let $B = \{x : S_x$ is covered by a finite subcover of $\mathcal{F}$.

Since S is closed and bounded, S has a minimum. Let $d = \min S$. Then $S_d = \{d\}$ which is certainly covered by a finite subcover of $\mathcal{F}$. Thus $d \in B$, so $B \neq \emptyset$. If we can show that B is not bounded above then it will contain a number p s.t. $p > \sup S$. But then $S_p = S$, and since $p \in B$, we can conclude that S is compact.

To this end, suppose B is bounded above and let $m = \sup B$. We shall show $m \in S$ and $m \notin S$ both lead to contradictions.

If $m \in S$, then since $\mathcal{F}$ is an open cover of S, $\exists F_0 \in \mathcal{F}$ s.t. $m \in F_0$. Since F_0 is open, there exists an interval $[x_1, x_2]$ in F_0 s.t. $x_1 < m < x_2$.

Since $x_1 < m$ and $m = \sup B$, $\exists F_1, \ldots, F_K \in \mathcal{F}$ s.t. $(F_1 \cup \ldots \cup F_K)$ is an open cover of S_{x_1}. But then $(F_0 \cup \ldots \cup F_K)$ cover S_{x_2}, so $x_2 \in B$, which contradicts $m = \sup B$.

If $m \notin S$, then since S is closed $\exists \varepsilon > 0$ s.t. $N(m;\varepsilon) \cap S = \emptyset$. But then $S_{m-\varepsilon} = S_{m+\varepsilon}$. Since $m - \varepsilon \in B$, we have $m + \varepsilon \in B$, which again contradicts $m = \sup B$.

Since the possibility that B is bounded above leads to a contradiction, we must conclude that B is not bounded above. Hence S is compact. $\blacksquare$

Theorem 45: (Bolzano-Weierstrass) If a bounded subset S of $\mathbb{R}$ contains infinitely many points, then $\exists$ at least one point in $\mathbb{R}$ that is an accumulation point of S.

i.e. The only sets w/o accumulation points are finite sets and infinite unbounded sets.

Pf: Let S be an infinite bounded subset of $\mathbb{R}$. Suppose S has no accumulation points. Then S is closed by Theorem 43, so by the Heine-Borel Theorem, S is compact. Since S has no accumulation points, given any $x \in S$ $\exists$ nbhd $N(x)$ of x s.t. $N(x) \cap S = \{x\}$. Now $\{N(x) : x \in S\}$ is an open cover of S, so $\exists$ a finite subcover $\{N(x_1), N(x_2), \ldots, N(x_n)\}$ which covers S. But $S \cap [N(x_1) \cup \cdots \cup N(x_n)] = \{x_1, \ldots, x_n\}$, which contradicts S has infinitely many points. $\blacksquare$

Theorem 46: Let $\mathcal{F} = \{K_\alpha : \alpha \in A\}$ be a family of compact subsets of $\mathbb{R}$. Suppose that the intersection of any finite subfamily of $\mathcal{F}$ is nonempty. Then $\bigcap \{K_\alpha : \alpha \in A\} \neq \emptyset$.

Pf: For each $\alpha \in A$, let $F_\alpha = \mathbb{R} \setminus K_\alpha$. Choose a member $K_1 \in \mathcal{F}$, and suppose no point of K_1 belongs to every K_α, i.e. $K_1 \cap (\bigcap K_\alpha) =$ Then the sets F_α form an open cover of K_1. Since K_1 is compact, $\exists$ finitely many $\alpha_1, \ldots, \alpha_n$ s.t. $K_1 \subseteq (F_{\alpha_1} \cup \ldots F_{\alpha_n})$. But this implies $K_1 \cap K_{\alpha_1} \cap \ldots \cap K_{\alpha_n} = \emptyset$, a contradiction. Thus some point of K_1 belongs to each K_α, and $\bigcap \{K_\alpha : \alpha \in A\} \neq$ $\blacksquare$

Corollary 2: (Nested Intervals Theorem) Let $\mathcal{F} = \{A_n : n \in \mathbb{N}\}$ be a family of closed bounded intervals in $\mathbb{R}$ s.t. $A_{n+1} \subseteq A_n \, \forall n \in \mathbb{N}$. Then $\bigcap_{n=1}^{\infty} A_n \neq \emptyset$.

Pf: Given any $n_1 < n_2 < \ldots < n_k$ in $\mathbb{N}$, we have $\bigcap_{n=1}^{k} A_n = A_{n_k} \neq \emptyset$. Thus Theorem 46 $\Rightarrow \bigcap_{n=1}^{\infty} A_n \neq \emptyset$. $\blacksquare$