§2.4: Cardinality

"Size" of a set = Cardinality

How can we compare the sizes of two sets? If $S = \{x \in \mathbb{R} \mid x^2 = 9\}$, then $S = \{-3, 3\}$ and we say S has 2 elements. If $T = \{1, 7, 11\}$ then T has 3 elements, and T is "larger" then S. How do we compare infinite sets like $\mathbb{N}$ and $\mathbb{R}$?

Def: Two sets S and T are <u>equinumerous</u>, $S \sim T$, if there exists a bijective function from S onto T.

Practice: If $\mathcal{F}$ is a family of sets, then equinumerous is a relation on $\mathcal{F}$. Show it is an equivalence relation.

Def: A set S is <u>finite</u> if $S = \emptyset$ or if $\exists n \in \mathbb{N}$ and a bijection $f: \{1, \ldots, n\} \to S$. If a set is not finite, it is <u>infinite</u>.

Notation: $I_n = \{1, 2, \ldots, n\}$. If $S \sim I_n$, S has n elements.
The "size" of $\emptyset$ is 0, and if the "size" of a set is not finite it is <u>transfinite</u>.

Practice: Suppose S and T are sets having n elements. Then from the definition $\exists\, g, h$ bijections, $g: I_n \to S$, $h: I_n \to T$. Show S and T are equinumerous directly by finding a bijection $f: S \to T$.
$(h \circ g^{-1}): S \to T$

Def: A set S is <u>denumerable</u> if $\exists$ bijection $f: \mathbb{N} \to S$. If a set is finite or denumerable, it is <u>countable</u>. If a set is not countabl it is <u>uncountable</u>.

The "size" of a countable set is $\aleph_0$ (aleph)

Note: Not all infinite sets are $\aleph_0$. There are different "sizes" of infin

The "size" of a set S is the cardinality of a set, denoted $|S|$.

Ex: Let $E = \{$even natural numbers$\}$. How does $|E|$ compare to $|\mathbb{N}| = \aleph_0$? Since $E \subset \mathbb{N}$, we would think $|E| < |\mathbb{N}|$, but actually $f: \mathbb{N} \to E$ given by $f(n) = 2n$ is a bijection of $\mathbb{N}$ onto E, so $|E| = |\mathbb{N}| = \aleph_0$.

Practice: Find a bijection $f: \mathbb{N} \to \mathbb{Z} \Rightarrow \mathbb{Z}$ is denumerable.

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ -\left(\frac{n-1}{2}\right) & \text{if } n \text{ is odd} \end{cases}$$

$f(1) = s_1 = 0$
$f(2) = s_2 = 1$
$f(3) = s_3 = -1$
$f(4) = s_4 = 2$
$\vdots$

Let S be a nonempty set which is finite. Then there is a bijection $f: I_n \to S$ which we can use to "count" the members of S; $f(1), f(2), \dots, f(n)$. If we let $f(k) = s_k$, for $1 \le k \le n$, we have $S = \{s_1, s_2, \dots, s_k\}$. This process works for any denumerable set.

Axiom 1: (Well ordering Property of $\mathbb{N}$) If $S \subseteq \mathbb{N}$, $S \neq \emptyset$, then S has a least element. i.e. $\exists m \in S$ s.t. $m \le k$ $\forall k \in S$.

Theorem 15: Let S be a countable set and let $T \subseteq S$. Then T is countable.

Pf: If T is finite, we are done. So suppose T is infinite $\Rightarrow$ S is infinite. $\exists f: \mathbb{N} \to S$ which is bijective, so we can write $S = \{s_1, s_2, \dots\}$. Let $A = \{n \in \mathbb{N} \mid s_n \in T\}$. Since A is a nonempty subset of $\mathbb{N}$, it has a least member, say a_1. Similarly $A \setminus \{a_1\}$ has a least member, a_2. In general, having chosen $\{a_1, a_2, \dots, a_k\}$, let a_{k+1} be the least member of $A \setminus \{a_1, a_2, \dots, a_k\}$.

Now define $g: \mathbb{N} \to \mathbb{N}$ by $g(n) = a_n$. Since T is infinite, g is defined for all $n \in \mathbb{N}$. Since $a_{n+1} \notin \{a_1, \dots, a_n\}$, g is injective. Thus the composition $f \circ g$ is injective. Since each element of T is somewhere in the list $s_1, s_2, \dots$ of S, $g(\mathbb{N})$ includes all the subscripts of terms in T. Thus $f \circ g$ is a bijection from $\mathbb{N}$ onto $T \Rightarrow T$ is denumerable. $\blacksquare$

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Theorem 16: Let S be a nonempty set. TFAE:

1) S is countable
2) $\exists f: S \to \mathbb{N}$ injective
3) $\exists g: \mathbb{N} \to S$ surjective

Pf:
($1 \Rightarrow 2$) Let S be countable. Then $\exists$ $h: I_n \to S$ for some $n \in \mathbb{N}$ if S is finite or $h: \mathbb{N} \to S$ if S is denumerable. In either case, $h^{-1}: S \to \mathbb{N}$ is an injection.

($2 \Rightarrow 3$) Suppose $\exists f: S \to \mathbb{N}$ which is injective. Then f is a bijection from S to $f(S)$, and f^{-1} is a bijection from $f(s)$ to S. Let $p \in S$. Define $g: \mathbb{N} \to S$ by

$$g(n) = \begin{cases} f^{-1}(n) & \text{if } n \in f(S) \\ p & \text{if } n \notin f(S) \end{cases}$$

Then $g[f(S)] = f^{-1}(f(S)) = S$ and $g[\mathbb{N} \setminus f(S)] = \{p\}$ so g is a surjection from $\mathbb{N}$ onto S.

($3 \Rightarrow 1$) Let $g: \mathbb{N} \to S$ be a surjection. Define $h: S \to \mathbb{N}$ by $h(s) =$ smallest $n \in \mathbb{N}$ s.t. $g(n) = s$. Then $h: S \to \mathbb{N}$ is injective, and bijective from S to $h(S) \subseteq \mathbb{N}$. Since $\mathbb{N}$ is countable, our previous theorem tells us $h(S)$ is countable. Since S and $h(S)$ are equinumerous, S is also countable. ∎

Ex: a) Let S and T be nonempty countable sets. Then $S \cup T$ is countable. We have (by the theorem) $f: \mathbb{N} \to S$, $g: \mathbb{N} \to T$ which are surjective. Define $h: \mathbb{N} \to S \cup T$ by

$$h(n) = \begin{cases} f\left(\frac{n+1}{2}\right) & \text{if } n \text{ is odd} \\ g\left(\frac{n}{2}\right) & \text{if } n \text{ is even} \end{cases}$$

h is surjective, so $S \cup T$ is countable.

b) Let S and T be nonempty countable sets. We will show $S \times T$ is countable, by using the factorization of numbers is unique, up to order.

$$\exists f : S \to \mathbb{N}, \exists g : T \to \mathbb{N} \quad \text{injections.}$$

Define $h : S \times T \to \mathbb{N}$ by

$$h(s,t) = 2^{f(s)} 3^{g(t)} \quad \text{where } s \in S,\ t \in T.$$

Then h is injective, for if $h(s,t) = h(u,v)$

$$\Rightarrow 2^{f(s)} 3^{g(t)} = 2^{f(u)} 3^{g(v)} \Rightarrow f(s) = f(u),\ g(t) = g(v)$$

$$\Rightarrow s = u,\ t = v.$$

$\therefore$ $S \times T$ is countable.

c) $\mathbb{Q}$ is countable.

First we will show $\mathbb{Q}^+ = \{x \in \mathbb{Q} \mid x > 0\}$ is countable.

Every $x \in \mathbb{Q}^+$ can be written as $\frac{m}{n}$ with $m, n \in \mathbb{N}$, $n \neq 0$ and m and n are relatively prime.

Define $f : \mathbb{Q}^+ \to \mathbb{N}$ by $f\left(\frac{m}{n}\right) = 2^m 3^n$.

f is injective, so $\mathbb{Q}^+$ is countable. The mapping $g : \mathbb{Q}^+ \to \mathbb{Q}^-$ given by $g(r) = -r$ is clearly bijective, so $\mathbb{Q}^+$ and $\mathbb{Q}^-$ are equinumerous, so $\mathbb{Q}^-$ is countable. Since $\mathbb{Q} = \mathbb{Q}^+ \cup \{0\} \cup \mathbb{Q}^-$, by applying (a) twice, $\mathbb{Q}$ is countable.

d) The union of a countable family of countable sets is countable. Let $\mathfrak{A} = \{S_1, S_2, \dots\}$.

$$\begin{array}{llll} S_1 : & S_{11} & S_{12} & S_{13} \ \dots \\ S_2 : & S_{21} & S_{22} & S_{23} \ \dots \\ S_3 : & S_{31} & S_{32} & S_{33} \ \dots \\ \vdots & & & \end{array}$$

Theorem 17: $\mathbb{R}$ is uncountable.

Pf: Uses the "diagonal process" of Cantor.

It suffices to show $J=(0,1)$ is uncountable.

Suppose J is countable, $J=\{x_1, x_2, x_3, \ldots\}=\{x_n \mid n\in\mathbb{N}\}$.

We shall construct a number $y\in(0,1)$ but $y\notin\{x_1,x_2,\ldots\}$.

Each element in J has an infinite decimal expansion, so we write

$$x_1 = 0.a_{11}\,a_{12}\,a_{13}\ldots$$
$$x_2 = 0.a_{21}\,a_{22}\,a_{23}\ldots$$
$$x_3 = 0.a_{31}\,a_{32}\,a_{33}\ldots$$
$$\vdots$$
$$x_n = 0.a_{n1}\,a_{n2}\,a_{n3}\ldots$$

where each $a_{ij}\in\{0,1,2,\ldots,9\}$. (Some numbers have more then one representation such as $0.5000\ldots$ and $0.4999\ldots$, but that's not a problem).

Let $y=0.b_1b_2b_3\ldots$ where

$$b_n=\begin{cases}2 & \text{if } a_{nn}\neq 2\\ 3 & \text{if } a_{nn}=2.\end{cases}$$

So $y\in J$, but $y\neq x_i$ for any i. This contradicts our assumption that J is countable, so J is uncountable.

Let's return to our question of comparing the "size" of sets. $|S|=|T|$ iff S and T are equinumerous iff $\exists$ bijection $f: S\to T$.

We define $|S|\le|T|$ if $\exists$ injection $f: S\to T$.

Theorem 18: Let S, T and U be sets.

a) If $S\subseteq T$, then $|S|\le|T|$

b) $|S|\le|S|$

c) If $|S|\le|T|$ and $|T|\le|U|$, then $|S|\le|U|$.

d) If $m,n\in\mathbb{N}$, $m\le n$, then $|\{1,2,\ldots,m\}|\le|\{1,2,\ldots,n\}|$.

e) If S is finite, then $|S|\le\aleph_0$

It is customary to denote $|\mathbb{R}|$ by c, for continuum. Since $\mathbb{Q} \subseteq \mathbb{R}$, we have $\aleph_0 \leq c$. Therefore $\aleph_0$ and c are unequal transfinite cardinals. There are others.

Notation: Given any set S, Let $\mathcal{P}(S)$ denote the collection of all subsets of S. This is called the <u>power set</u> of S.

Theorem 19: For any set S, we have $|S| < |\mathcal{P}(S)|$.

Pf: The function $g: S \to \mathcal{P}(S)$ given by $g(s) = \{s\}$ is clearly injective, So $|S| \leq |\mathcal{P}(S)|$. To prove $|S| \neq |\mathcal{P}(S)|$, we show that no function from S to $\mathcal{P}(S)$ can be surjective. Suppose that $f: S \to \mathcal{P}(S)$. Then for each $x \in S$, $f(x) \subseteq S$. Now for some $x \in S$ it may be that $x \in f(x)$ and for others it may not be. Let

$$T = \{x \in S \mid x \notin f(x)\}$$

Now $T \subseteq S$, So $T \in \mathcal{P}(S)$. If f were surjective, then $T = f(y)$ for some $y \in S$. Now either $y \in T$ or $y \notin T$, but both possibilities lead to contradictions! If $y \in T$, then $y \notin f(y)$, by how T is defined But $f(y) = T$, so $y \notin T$. On the other hand if $y \notin T$, then $y \notin f(y)$ $\Rightarrow y \in T$.

Thus no function from S to $\mathcal{P}(S)$ can be surjective, so $|S| \neq |\mathcal{P}(S)|$. ⊠

By applying Theorem 19 over and over again, we get

$$\aleph_0 = |\mathbb{N}| < |\mathcal{P}(\mathbb{N})| < |\mathcal{P}(\mathcal{P}(\mathbb{N}))| < \ldots$$

It can be shown that $|\mathcal{P}(\mathbb{N})| = c$

Is there any set "in between" $\mathbb{N}$ and $\mathbb{R}$ in size? i.e. $\exists$ a cardinal number λ s.t. $\aleph_0 < \lambda < c$?

This is the <u>continuum hypothesis</u>. Famous unsolved problem.