Show all work. Please box your answers. Be sure to write in complete sentences when appropriate. Also, I prefer exact answers like $\sqrt{2}$ instead of 1.414. Note that a symbol $\square$ indicates that graph paper might be useful for that problem.

Cylindrical and Spherical Coordinates
1. $\square$ Sketch the graph of the given cylindrical or spherical equation.
   (a) $r = 5$
   (b) $\theta = \frac{\pi}{6}$
   (c) $\rho = 3 \cos \phi$
   (d) $r^2 \cos^2 \theta + z^2 = 4$

2. Make the required change
   (a) $x^2 - y^2 - z^2 = 1$ to spherical coordinates
   (b) $x + y = 4$ to cylindrical coordinates

Functions of Two or More Variables (revisited)
3. $\square$ For the following functions, sketch a contour diagram with at least four labeled contours. Describe in words the contours and how they are spaced.
   (a) $f(x, y) = x + y$
   (b) $g(x, y) = 2x - y$
   (c) $h(x, y) = xy$
   (d) $k(x, y) = x^2 + y^2$

4. The figure below shows a contour map of a hill with two paths, $A$ and $B$.

(a) On which path, $A$ or $B$, will you have to climb more steeply?
(b) On which path, $A$ or $B$, will you probably have a better view of the surrounding countryside? (Assuming trees do not block your view.)
(c) Near which path is there more likely to be a river?
5. For each of the surfaces below, draw a possible contour diagram, marked with reasonable $z$ values. (Note: there are many possible answers.)

(a)  

(b)  

(c)  

6. Identify the contour diagrams and the surfaces below corresponding to the following equations. Assume that each contour diagram is drawn in a square window.

(a) $z = \sin x$

(b) $z = xy$

(c) $z = e^{-(x^2+y^2)}$

(d) $z = 1 - 2x - y$

(e) $z = x^2 + 4y^2$
7. Hot water is entering a rectangular swimming pool at the surface of the pool in one corner. Sketch a possible contour diagram for the temperature of the pool at the surface and three feet below the surface.

8. A level surface for a function $f$ of three variables is the graph of the set of points in three-dimensional space whose coordinates satisfy an equations $f(x, y, z) = k$ where $k$ is a constant. Describe geometrically the level surfaces for the function $f(x, y, z) = 16x^2 + 16y^2 - 9z^2$, $k \in \mathbb{R}$.

9. Match the following functions with the level surfaces below.

(a) $f(x, y, z) = y^2 + z^2$.
(b) $f(x, y, z) = x^2 + z^2$. 
10. The figure below shows contour diagrams of temperature in degrees Fahrenheit in a room at three different times. Describe in as much detail as you can the heat flow in the room. What could be causing this?

![Contour Diagrams](image)

11. Describe the surface $x^2 + y^2 = (2 + \sin z)^2$. In general, if $f(z) > 0$ for all $z$, describe the surface $x^2 + y^2 = (f(z))^2$. [Hint: Think “Pottery”]

12. Sketch the graph of $f(x, y) = \sqrt{16 - 4x^2 - y^2}$.

13. Sketch the level curve $z = k$ for $z = \frac{x}{y}$, $k = -2, -1, 0, 1, 2$.

**Partial Derivatives**

14. The monthly mortgage payment in dollars, $P$, for a house is a function of three variables $P = f(A, r, N)$ where $A$ is the amount borrowed in dollars, $r$ is the interest rate, and $N$ is the number of years before the mortgage is paid off.

(a) Suppose $f(92000, 14, 30) = 1090.08$. What does this tell you in financial terms?

(b) Suppose $\frac{\partial P}{\partial r}(92000, 14, 30) = 72.82$. What is the financial significance of the number 72.82?

(c) Would you expect $\frac{\partial P}{\partial N}$ to be positive or negative? Why?

15. A drug is injected into a patient’s blood vessel. The function $c = f(x, t)$ represents the concentration of the drug at a distance $x$ in the direction of the blood flow measured from the point of injection and at time $t$ since the injection. What are the units of the following partial derivatives? What are their practical interpretations? What do you expect their signs to be?

(a) $\frac{\partial c}{\partial x}$

(b) $\frac{\partial c}{\partial t}$

16. Suppose that $c$ represents the cardiac output, which is the volume of blood flowing through a person’s heart, and that $s$ represents the systemic vascular resistance (SVR), which is the resistance to blood flowing through veins and arteries. Let $p$ be a person’s blood pressure. Then $p = f(c, s)$ is a function of $c$ and $s$. 
(a) What does $\frac{\partial p}{\partial c}$ represent?

(b) Suppose $p = kcs$, where $k$ is a positive constant. Sketch the level curves of $p$. What do they represent? Label your axes.

(c) For a person with a weak heart, it is desirable to have the heart pumping against less resistance, while maintaining the same blood pressure. Such a person is given the drug Nitroglycerine to decrease the SVR and the drug Dopamine to increase the cardiac output. Represent this on a graph showing level curves (use the same function $p = kcs$). Put a point $A$ on the graph representing the person’s state before the drugs are given and a point $B$ for after.

(d) Right after a heart attack, a patient’s cardiac output drops, thereby causing the blood pressure to drop. A common mistake made by medical residents is to get the patient’s blood pressure back to normal by using drugs to increase the SVR, rather than by increasing the cardiac output. On a graph of the level curves of $p$, put a point $D$ representing the patient before the heart attack, and a point $E$ representing the patient right after the heart attack, and a third point $F$ representing the patient after the resident has given the drugs to increase the SVR.

17. Find the indicated partial derivatives. Assume the variables are restricted to a domain on which the function is defined.

(a) $\frac{\partial A}{\partial h}$ if $A = \frac{1}{2}(a + b)/h$

(b) $F_v$ if $F = \frac{mv^2}{r}$

(c) $\frac{\partial}{\partial m} \left( \frac{1}{2}mv^2 \right)$

(d) $u_E$ if $u = \frac{1}{2}v_0E^2 + \frac{1}{2\mu_0}B^2$

(e) $\frac{\partial F}{\partial m}$ if $F = \frac{Gm_1m_2}{r^2}$

(f) $\frac{\partial}{\partial t} \left( v_0t + \frac{1}{2}at^2 \right)$

(g) $z_y$ if $z = \frac{3x^2y^7 - y^2}{15xy - 8}$

(h) $\frac{\partial m}{\partial v}$ if $m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}$

(i) $\frac{\partial \alpha}{\partial \beta}$ if $\alpha = \frac{\exp(x\beta - 3)}{2y\beta + 5}$

18. Suppose you know that

$$f_x(x, y) = 4x^3y^2 - 3y^4,$$

$$f_y(x, y) = 2x^4y - 12xy^3.$$ Can you find a function $f$ which has these partial derivatives? If so, are there any others?

19. A function of two variables that satisfies Laplace’s Equation,

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

is said to be harmonic. Show that the function $f(x, y) = \ln(4x^2 + 4y^2)$ is a harmonic function.

20. A bee was flying upward along the curve that is the intersection of $z = x^4 + xy^3 + 12$ with the plane $x = 1$. At the point $(1, -2, 5)$, it went off on the tangent line. Where did the bee hit the $xz$-plane?

21. Calculate all four second-order partial derivatives and show that $z_{xy} = z_{yx}$. 
(a) \( z = \sin \left( \frac{x}{y} \right) \)
(b) \( z = xe^y \)
(c) \( z = x^y \)
(d) \( z = \ln(xy) \)

22. If \( z = f(x) + yg(x) \) what can you say about \( z_{yy} \)? Why?

23. If \( z_{xy} = 4y \), what can you say about the value of
   (a) \( z_{yx} \)?
   (b) \( z_{xyx} \)?
   (c) \( z_{xyy} \)?

**Limits and Continuity**

24. Find the indicated limits, or state that they do not exist.
   (a) \( \lim_{(x,y) \to (-2,1)} (xy^3 - xy + 3y^2) \)
   (b) \( \lim_{(x,y) \to (0,0)} \frac{x^2 + y^2}{x^4 - y^4} \)

25. Describe the largest set for which the following functions are continuous.
   (a) \( f(x, y) = \ln(1 - x^2 - y^2) \)
   (b) \( f(x, y) = \begin{cases} \frac{\sin(xy)}{xy} & : \ xy \neq 0 \\ 1 & : \ xy = 0 \end{cases} \)

26. Show that
   \[ \lim_{(x,y) \to (0,0)} \frac{xy + y^3}{x^2 + y^2} \]
   does not exist.

27. Let
   \[ f(x, y) = \begin{cases} \frac{x^2 - 4y^2}{x - 2y} & : \ x \neq 2y \\ \frac{x^2}{g(x)} & : \ x = 2y \end{cases} \]

   If \( f \) is continuous in the whole plane, find a formula for \( g(x) \).

28. Occasionally it is easier to analyze the continuity of \( f(x, y) \) by changing to polar coordinates. Let \( f(x, y) = \frac{x^2 - y^2}{x^2 + y^2} \) for \( (x, y) \neq (0, 0) \) and \( f(0, 0) = 0 \). In polar coordinates,
   \[ f(r, \theta) = \frac{r^2 \cos^2 \theta - r^2 \sin^2 \theta}{r^2} = \cos 2\theta \]
   which takes all values between -1 and 1 in every neighborhood of the origin. Therefore, \( \lim_{(x,y) \to (0,0)} f(x, y) \) does not exist and \( f \) is not continuous at the origin. Let each of the following functions have the value 0 at \((0,0)\). Which of them are continuous at \((0,0)\), and which are discontinuous there?
   (a) \( f(x, y) = \frac{xy}{\sqrt{x^2 + y^2}} \)
   (b) \( f(x, y) = \frac{xy}{x^2 + y^2} \)
   (c) \( f(x, y) = \frac{x^{7/3}}{x^2 + y^2} \)
(d) \( f(x, y) = xy \frac{x^2 - y^2}{x^2 + y^2} \)
(e) \( f(x, y) = \frac{x^2 y^2}{x^2 + y^2} \)
(f) \( f(x, y) = \frac{xy^2}{x^2 + y^2} \)

29. The French Railroad. Suppose that Paris is located at the origin of the \( xy \)-plane. Rail lines emanate from Paris along all rays, and these are the only rail lines. Determine the set of discontinuities of the following functions.

(a) \( f(x, y) \) is the distance from \((x, y)\) to \((1,0)\) on the French railroad.
(b) \( g(u, v, x, y) \) is the distance from \((u, v)\) to \((x, y)\) on the French railroad.

Differentiability and the Differential

30. If \( z = e^{3x} \sin y \), find \( dz \).

31. Find the differentials to the following functions:

(a) \( f(x, y) = \cos(x^2 y) \).
(b) \( g(s, t) = s^4 + ts^2 \).
(c) \( \theta(\alpha, \beta) = e^{0.1\alpha} \cos(2\pi\beta + 10\alpha) \).

32. Find the differentials of the functions in Problem 31 at the points:

(a) \((x, y) = (1, 2)\).
(b) \((s, t) = (3, 5)\).
(c) \((\alpha, \beta) = (\pi, 2)\).

33. One mole of ammonia gas is contained in a vessel which is capable of changing its volume (e.g. a piston). The total energy \( U \) (in Joules) of the ammonia is a function of the volume \( V \) (in m\(^3\)) of the container, and the temperature \( T \) (in Kelvin) of the gas. The differential \( dU \) is given by

\[
\frac{dU}{dV} = 840 \quad \text{and} \quad \frac{dU}{dT} = 27.32.
\]

(a) How does the energy change if the volume is held constant and the temperature is increased slightly?
(b) How does the energy change if the temperature is held constant and the volume is increased slightly?
(c) Find the approximate change in energy if the gas is compressed by 100 cm\(^3\) and heated by 2 K.

Directional Derivatives

34. Use a difference quotient to estimate the rate of change of \( f(x, y) = xe^y \) at the point \((1,1)\) as you move in the direction of the vector \( \hat{\imath} + 2\hat{j} \).

35. Draw a contour diagram for the function \( z = f(x, y) = x^2 + 4y^2 \) on the window \(-2 \leq x \leq 2\) and \(-2 \leq y \leq 2\) with the contours \( z = 0, z = 1, z = 2, z = 3, z = 4 \). Try to be as accurate as possible. Use your contour diagram to estimate each of the following directional derivatives.

(a) \( f_u(0, 1) \).
(b) \( f_v(0, 1) \).
(c) \( f_u(1, 1) \), where \( \vec{u} = (\hat{\imath} - \hat{j})/\sqrt{2} \).
(d) \( f_v(1, 1) \), where \( \vec{v} = (\hat{\imath} + \hat{j})/\sqrt{2} \).
36. Evaluate analytically the directional derivatives from Problem 35.

37. Find the directional derivative of \( f(x, y, z) = x^2y + y^2z \) at the point \((1, 2, 3)\) in the direction of the following vectors (note that they are not unit vectors):
   (a) \( \vec{u} = -\vec{j} + \vec{k} \).
   (b) \( \vec{v} = 2\vec{i} + 3\vec{j} + 5\vec{k} \).

### The Gradient

38. Suppose that \( f \) is differentiable at \((a, b)\). Then there is always a direction in which the rate of change of \( f \) at \((a, b)\) is 0. True or false? Explain your answer.

39. Compute the gradient of the given functions.
   (a) \( f(x, y) = x^3 + y^2 \)
   (b) \( g(u, v) = uv \)
   (c) \( h(x, t) = e^{\alpha t} \sin \left( \frac{\pi x}{L} \right) \)

40. Find \( \nabla z \) at the specified point.
   (a) \( z = \sin(x/y) \), at \((\pi, 1)\)
   (b) \( z = \frac{xe^y}{x^2 + y^2} \), at \((1, 2)\)

41. Let \( f(x, y) = \frac{x+y}{x+y} \) and let \( A = (2, 3) \). Find the directional derivative at \( A \) in the direction of the vectors
   (a) \( \vec{v} = 3\vec{i} - 2\vec{j} \),
   (b) \( \vec{u} = -\vec{i} + 4\vec{j} \).
   (c) What is the direction of greatest increase at \( A \)?

42. The temperature at any point in the plane is given by the function
   \[ T(x, y) = \frac{100}{x^2 + y^2 + 1} \]
   (a) Where on the plane is it hottest? What is the temperature at that point?
   (b) Find the direction of the greatest increase in the temperature at the point \((3, 2)\). What is the magnitude of that greatest increase?
   (c) Find the direction of the greatest decrease in temperature at the point \((3, 2)\).
   (d) Is the direction of the vector you found in part (b) towards the point you found in part (a)?
   (e) Find a direction at the point \((3, 2)\) in which the temperature does not increase or decrease (See Problem 38).
   (f) What shape are the level curves of \( T \)?

43. Sketch the level curves of the function \( z = f(x, y) = \sqrt{xy} \) on the window \( 0 \leq x \leq 5 \) and \( 0 \leq y \leq 5 \). You should have the contours \( z = 1 \), \( z = 2 \), \( z = 3 \), and \( z = 4 \). At the points \((1, 1)\) and \((1, 4)\) on your sketch, draw a vector representing \( \nabla f \) (no computations required). Explain how you decided the approximate length and direction of each vector.

44. **Extra Credit** (Do this problem only if you have finished all of the other problems on this homework).
   Consider \( S \) to be the surface represented by the equation \( F = 0 \), where
   \[ F(x, y, z) = x^2z^2 - y. \]
   (a) Find all points on \( S \) where a normal vector is parallel to the xy-plane.
   (b) Find the tangent plane to \( S \) at the points \((0, 0, 1)\) and \((1, 1, 1)\).
   (c) Find the unit vectors \( \vec{u}_1 \) and \( \vec{u}_2 \) pointing in the direction of maximum increase of \( F \) at the points \((0, 0, 1)\) and \((1, 1, 1)\) respectively.