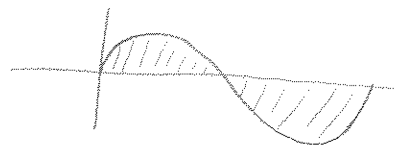
Final Review Solutions

1. If f Is A Positive Function The Definite Integral Is The Same As Finding The Area. If f Is Not Strictly A Positive Function Then To Find The Area, $\text{Area} = \int_{R^+} f\,dx - \int_{R^-} f\,dx$ Where $R^+ = \{x \in [a,b] \mid f(x) \geq 0\}$, $R^- = \{x \in [a,b] \mid f(x) < 0\}$ Where $\int_a^b f\,dx = \int_{R^+} f\,dx + \int_{R^-} f\,dx$, So Area $\neq$ Definite Integral.

Ex: $y = \sin x \qquad 0 \leq x \leq 2\pi \qquad \int_0^{2\pi} \sin x\,dx = \boxed{0}$



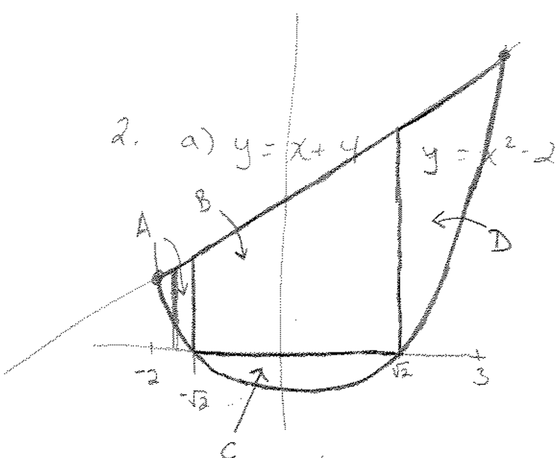
But Area Enclosed By $y = \sin x$

Is $A = \int_0^{\pi} \sin x\,dx - \int_{\pi}^{2\pi} \sin x\,dx$

$= -\cos x \Big|_0^{\pi} - (-\cos x)\Big|_{\pi}^{2\pi}$

$= -(-1-1) + (1-(-1)) = -(-2) + (2) = \boxed{4}$

2. a)



$x^2 - 2 = x + 4$

$x^2 - x - 6 = 0$

$(x-3)(x+2) = 0$

$x = 3, -2$

$0 = x^2 - 2$

$x^2 = 2$

$x = \pm\sqrt{2}$

Area = A + B + C + D

$$A = \int_{-2}^{3} x + 4 - (x^2 - 2)\,dx = \int_{-2}^{3} x + 6 - x^2\,dx = \frac{1}{2}x^2 + 6x - \frac{1}{3}x^3 \Big|_{-2}^{3}$$

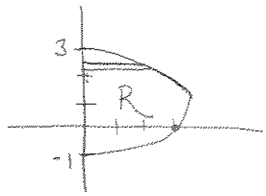
$$= \frac{1}{2}\left(3^2 - (-2)^2\right) + 6\left(3 - (-2)\right) - \frac{1}{3}\left(3^3 - (-2)^3\right)$$

$$= \frac{1}{2}(9-4) + 6(5) - \frac{1}{3}(27+8)$$

$$= \frac{5}{2} + 30 - \frac{35}{3}$$

$$\boxed{= \frac{125}{6}}$$

b) $x = (3-y)(y+1), \quad x = 0$



$A_\Delta = x \Delta y$

$$A = \int_{-1}^{3} (3-y)(y+1)\,dy$$

$$= \int_{-1}^{3} 2y + 3 - y^2 \, dy$$

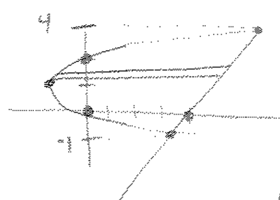
$$= y^2 + 3y - \tfrac{1}{3}y^3 \Big|_{-1}^{3}$$

$$= (3^2 - 1) + 3(3-(-1)) - \tfrac{1}{3}(3^3 - (-1)^3)$$

$$= 8 + 12 - \tfrac{1}{3}(27+1) = 20 - \tfrac{28}{3} = \boxed{\tfrac{32}{3}}$$

c) $x = y^2 - 2y, \quad x - y - 4 = 0$

$x = y(y-2) \qquad x = y + 4$



$y^2 - 2y = y + 4$

$y^2 - 3y - 4 = 0$

$(y-4)(y+1) = 0$

$y = 4, -1$

$$A = \int_{-1}^{4} y + 4 - (y(y-2))\,dy$$

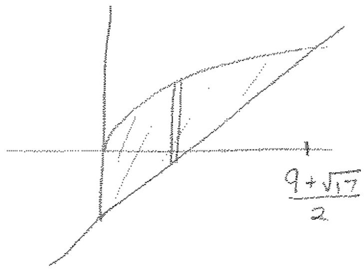
$$= \int y + 4 - y^2 + 2y \, dy$$

$$= \int 3y + 4 - y^2 dy = \tfrac{3}{2}y^2 + 4y - \tfrac{1}{3}y^3 \Big|_{-1}^{4}$$

$$= \tfrac{3}{2}(16-1) + 4(5) - \tfrac{1}{3}(4^3+1) = \tfrac{45}{2} + 20 - \tfrac{65}{3}$$

$$= \frac{135 + 120 - 130}{6} = \boxed{\frac{125}{6}}$$

d) $y=\sqrt{x}\ ,\ y=x-4\ ,\ x=0$



$$A=\int_0^{\frac{9+\sqrt{17}}{2}} \sqrt{x}-(x-4)\,dx$$

$$=\int x^{1/2}-x+4\,dx$$

$$=\frac{2}{3}x^{3/2}-\frac{1}{2}x^2+4x\ \Big|_0^{\frac{9+\sqrt{17}}{2}}$$

$$\boxed{=\frac{2}{3}\left(\frac{9+\sqrt{17}}{2}\right)^{3/2}-\frac{1}{2}\left(\frac{9+\sqrt{17}}{2}\right)^2+4\left(\frac{9+\sqrt{17}}{2}\right)}$$

$\sqrt{x}=x-4$

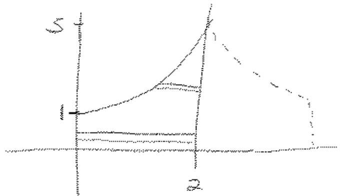
$x=x^2-8x+16$

$x^2-9x+16=0$

$x=\dfrac{9\pm\sqrt{81-64}}{2}=\dfrac{9\pm\sqrt{17}}{2}$

$(2-\sqrt{y-1})(2-\sqrt{y-1})$

3. $y=x^2+1\ ,\ x=0\ ,\ x=2$ REVOLVED ABOUT $x=2$



$$V=\int_0^1 \pi(2)^2\,dy+\int_1^5 \pi\left[2-(y-1)^{1/2}\right]^2 dy$$

$$=4\pi y\Big|_0^1+\pi\int 4-4\sqrt{y-1}+y-1\ dy$$

$$=4\pi+\pi\int 3-4\sqrt{y-1}+y\ dy$$

$$=4\pi+\pi\left(3y-\frac{8}{3}(y-1)^{3/2}+\frac{1}{2}y^2\right)\Big|_1^5$$

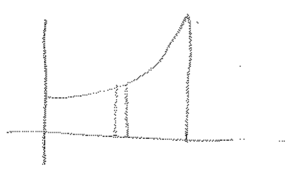
$$=4\pi+\pi\left(3(4)-\frac{8}{3}(8-0)+\frac{1}{2}(25-1)\right)=4\pi+\pi\left(12-\frac{64}{3}+12\right.$$

$$\boxed{=\frac{20\pi}{3}}$$

$y=x^2+1$

$y-1=x^2$

$x=\sqrt{y-1}$



$y=x^2+1$

$$V=\int_0^2 2\pi(2-x)(x^2+1)\,dx$$

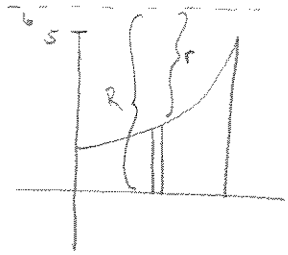
$$=2\pi\int 2x^2+2-x^3-x\ dx$$

$$=2\pi\left(\frac{2}{3}x^3+2x-\frac{1}{4}x^4-\frac{1}{2}x^2\right)\Big|_0^2$$

$$=2\pi\left(\frac{2}{3}(8)+4-\frac{1}{4}(16)-\frac{1}{2}(4)\right)$$

$$=2\pi\left(\frac{16}{3}+4-4-2\right)=2\pi\left(\frac{16-6}{3}\right)=2\pi\left(\frac{10}{3}\right)=\boxed{\frac{20\pi}{3}}$$

4. $y = x^2+1,\ x=0,\ x=2$ REVOLVED ABOUT $y=6$



$V_\Delta = \pi(R^2 - r^2)\Delta x$

$$V = \int_0^2 \pi\left(6^2 - (x^2+1)^2\right)dx$$

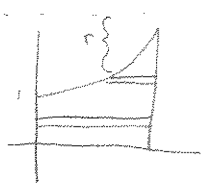
$$= \int_0^2 \pi\left[36 - (x^4+2x^2+1)\right]dx$$

$$= \pi\int 35 - x^4 - 2x^2\,dx \quad = \pi\left(35x - \tfrac{1}{5}x^5 - \tfrac{2}{3}x^3\right)\Big|_0^2$$

$$= \pi\left(35(2) - \tfrac{1}{5}(2)^5 - \tfrac{2}{3}(2)^3\right) = \pi\left(70 - \tfrac{32}{5} - \tfrac{16}{3}\right)$$

$$= \pi\left(\frac{1050-96-80}{15}\right)$$

$$\boxed{= \frac{874\pi}{15}}$$



$x = \sqrt{y-1}$

$V_\Delta = 2\pi(6-y)(2-\sqrt{y-1})\Delta y$

$$V = \int_0^1 2\pi(6-y)2\,dy + \int_1^5 2\pi(6-y)(2-\sqrt{y-1})\,dy$$

$$= 4\pi\left(6y - \tfrac{1}{2}y^2\right)\Big|_0^1 + 2\pi\int_1^5 12 - 6\sqrt{y-1} - 2y + y\sqrt{y-1}\ dy$$

$$= 4\pi\left(6-\tfrac{1}{2}\right) + 2\pi\left[12y - 4(y-1)^{3/2} - y^2\Big|_1^5 + \int_1^5 y\sqrt{y-1}\,dy\right]$$

$$= 22\pi + 2\pi\left[12(4) - 4(8) + 4(0) - 25 + 1\right] + 2\pi\int_1^5 y\sqrt{y-1}\,dy$$

$$= 22\pi + 2\pi\left[48 - 32 - 24\right] + 2\pi\int y\sqrt{y-1}\,dy$$

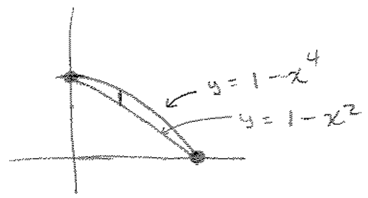
$$= 22\pi - 16\pi + 2\pi\int y\sqrt{y-1}\,dy$$

$$\boxed{6\pi + 2\pi\int_1^5 y\sqrt{y-1}\,dy}$$

5. $y = 1 - x^2$

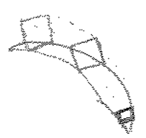
$y = 1 - x^4$

CROSSSECTIONS $\perp$ TO X-AXIS ARE SQUARES



$$V_\Delta = \left[(1-x^4)-(1-x^2)\right]^2 \Delta x$$

$$V = \int_0^1 (x^2-x^4)^2\, dx$$



$$= \int_0^1 x^4 - 2x^6 + x^8 \, dx$$

$$= \tfrac{1}{5}x^5 - \tfrac{2}{7}x^7 + \tfrac{1}{9}x^9 \Big|_0^1$$

$$= \frac{1}{5} - \frac{2}{7} + \frac{1}{9} = \frac{63-90+35}{313} = \boxed{\frac{8}{313}}$$

6. a) $y = \dfrac{x^4+3}{6x}$ $\quad x = 1,\ x = 3$

$$y = \tfrac{1}{6}x^3 + \tfrac{1}{2}x^{-1}$$

$$L = \int_1^3 \sqrt{1 + \left(\tfrac{1}{2}x^2 - \tfrac{1}{2}x^{-2}\right)^2}\, dx = \int \sqrt{1 + \tfrac{1}{4}x^4 - \tfrac{1}{2} + \tfrac{1}{4}x^{-4}}\, dx$$

$$= \int \sqrt{\tfrac{1}{4}(x^4+2+x^{-4})}\, dx = \tfrac{1}{2}\int \sqrt{(x^2+x^{-2})^2}\, dx$$

$$= \tfrac{1}{2}\int x^2 + x^{-2}\, dx = \tfrac{1}{2}\left(\tfrac{1}{3}x^3 - x^{-1}\right)\Big|_1^3$$

$$= \tfrac{1}{2}\left[\tfrac{1}{3}(27-1) - \left(\tfrac{1}{3}-1\right)\right] = \tfrac{1}{2}\left(\tfrac{26}{3}+\tfrac{2}{3}\right) = \tfrac{1}{2}\left(\tfrac{28}{3}\right) = \boxed{\tfrac{14}{3}}$$

b) $x = 4\sin t,\ y = 4\cos t - 5 \qquad 0 \le t \le \pi$

$$L = \int_0^{\pi} \sqrt{(4\cos t)^2 + (-4\sin t)^2}\ dt$$

$$= \int_0^{\pi} \sqrt{16(\cos^2 t + \sin^2 t)}\ dt = \int 4\,dt = 4t\Big|_0^{\pi} = \boxed{4\pi}$$

c) $30xy^3 - y^8 = 15 \qquad y = 1,\ y = 3$

$30xy^3 = 15 + y^8$

$x = \dfrac{15 + y^8}{30y^3}$

$$\frac{dx}{dy} = \frac{8y^7(30y^3) - 90y^2(15 + y^8)}{30^2 y^6}$$

$$= \frac{240y^{10} - 1350y^2 - 90y^{10}}{30^2 y^6}$$

$$= \frac{150y^{10} - 1350y^2}{30^2 y^6} = \frac{30y^2(5y^8 - 45)}{30^2 y^6}$$

$$= \frac{5y^8 - 45}{30y^4}$$

$$L = \int_1^3 \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\ dy$$

$$= \int_1^3 \sqrt{1 + \left(\frac{5y^8 - 45}{30y^4}\right)^2}\ dy$$

$$= \int_1^3 \sqrt{\frac{30^2y^8 + 25y^{16} - 450y^8 + 2025}{(30y^4)^2}}\ dy$$

$$= \int_1^3 \frac{1}{30y^4}\sqrt{25y^{16} + 450y^8 + 2025}\ dy$$

$$= \int_1^3 \frac{1}{30y^4}\sqrt{(5y^8 + 45)^2}\ dy = \int_1^3 \frac{1}{30y^4}(5y^8 + 45)\ dy$$

$$= \frac{1}{30}\int 5y^4 + 45y^{-4}\ dy = \frac{1}{30}\Big[y^5 - 15y^{-3}\Big]_1^3$$

$$\boxed{= \frac{1}{30}\Big[(3^5 - 1) - 15((3)^{-3} - 1)\Big]}$$

7. a) $y = \dfrac{x^6+2}{8x^2}$, $1 \le x \le 3$

$$A = \int_1^3 2\pi\left(\frac{x^6+2}{8x^2}\right)\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$\frac{dy}{dx} = \frac{6x^5(8x^2) - 16x(x^6+2)}{64x^4} = \frac{8x\left[6x^6 - 2x^6 - 4\right]}{64x^4} = \frac{4x^6-4}{8x^3} = \frac{x^6-1}{2x^3}$$

$$A = \int_1^3 2\pi\left(\tfrac{1}{8}x^4 + \tfrac{1}{4}x^{-2}\right)\sqrt{1+\left(\frac{x^6-1}{2x^3}\right)^2}\,dx$$

$$= \int 2\pi\left(\tfrac{1}{8}x^4 + \tfrac{1}{4}x^{-2}\right)\sqrt{\frac{4x^6+(x^6-1)^2}{4x^6}}\,dx$$

$$= \int 2\pi\left(\tfrac{1}{8}x^4 + \tfrac{1}{4}x^{-2}\right)\left(\frac{1}{2x^3}\right)\sqrt{4x^6 + x^{12} - 2x^6 + 1}\,dx$$

$$= \int 2\pi\left(\tfrac{1}{16}x + \tfrac{1}{8}x^{-5}\right)\sqrt{x^{12}+2x^6+1}\,dx$$

$$= \int 2\pi\left(\tfrac{1}{16}x + \tfrac{1}{8}x^{-5}\right)\sqrt{(x^6+1)^2}\,dx$$

$$= \int 2\pi\left(\tfrac{1}{16}x + \tfrac{1}{8}x^{-5}\right)(x^6+1)\,dx$$

$$= \frac{\pi}{8}\int (x + 2x^{-5})(x^6+1)\,dx = \frac{\pi}{8}\int x^7 + x + 2x + 2x^{-5}\,dx$$

$$= \frac{\pi}{8}\int x^7 + 3x + 2x^{-5}\,dx = \frac{\pi}{8}\left[\tfrac{1}{8}x^8 + \tfrac{3}{2}x^2 - \tfrac{1}{2}x^{-4}\right]\Big|_1^3$$

$$\boxed{= \frac{\pi}{8}\left[\tfrac{1}{8}(3^8-1) + \tfrac{3}{2}(8) - \tfrac{1}{2}(3^{-4}-1)\right]}$$

b) $y = \frac{1}{3}x^3$ $\quad 1 \le x \le \sqrt{7}$

$$A = \int_1^{\sqrt{7}} 2\pi\left(\tfrac{1}{3}x^3\right)\sqrt{1+(x^2)^2}\,dx = \frac{2\pi}{3}\int x^3\sqrt{1+x^4}\,dx \qquad \begin{matrix} u = 1+x^4 \\ du = 4x^3\,dx \end{matrix}$$

$$= \frac{2\pi}{3}\int_2^{50} x^3 u^{1/2}\,\frac{du}{4x^3} = \frac{\pi}{6}\int_2^{50} u^{1/2}\,du = \frac{\pi}{6}\left(\tfrac{2}{3}u^{3/2}\right)\Big|_2^{50}$$

$$\boxed{= \frac{\pi}{9}\left(50^{3/2} - 2^{3/2}\right)}$$

c) $x = 1 - t^2,\ y = 2t \qquad 0 \le t \le 1$

$$A = \int_0^1 2\pi(2t)\sqrt{(-2t)^2+(2)^2}\,dt$$

$$= 4\pi \int t\sqrt{4t^2+4}\,dt = 8\pi \int t\sqrt{t^2+1}\,dt \qquad u = t^2+1$$

$$du = 2t\,dt$$

$$= 8\pi \int_1^2 t\,u^{1/2}\,\frac{du}{2t} = 4\pi \int_1^2 u^{1/2}\,du = 4\pi\left(\tfrac{2}{3}u^{3/2}\right)\Big|_1^2$$

$$\boxed{= \frac{8\pi}{3}\left(2^{3/2}-1\right)}$$

8. 0.3 m REQUIRED 5 N. $W = ?$ WHEN STRETCHED 2m?

$F(x) = kx$

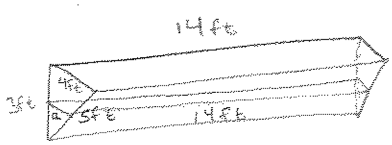
$5 = k(0.3) \Rightarrow k = \frac{5}{0.3}$

$F(x) = \frac{5}{0.3}x$

$$W = \int_0^2 \frac{5}{0.3}x\,dx = \frac{5}{0.3}\left(\tfrac{1}{2}x^2\right)\Big|_0^2$$

$$= \frac{5}{0.6}(4 = \frac{10}{0.3}\ N\cdot m = \boxed{\frac{100}{3}\ N\cdot m}$$

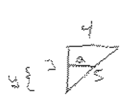
9.



$V_\Delta = a(14)\Delta y = 14\left(\frac{4y}{3}\right)\Delta y$

$W_\Delta = \left[14\left(\frac{4y}{3}\right)62.4\,\Delta y\right](3-y)$

$\delta = 62.4$ lb/ft³



$\frac{y}{3} = \frac{a}{4} \Rightarrow a = \frac{4y}{3}$

$$W = \int_0^3 \left[14\left(\frac{4y}{3}\right)62.4\right](3-y)\,dy$$

$$W = 14(62.4)\left(\frac{4}{3}\right)\int 3y - y^2\,dy$$

$$= 14(62.4)\left(\frac{4}{3}\right)\left[\frac{3}{2}y^2 - \frac{1}{3}y^3\right]_0^3 = \boxed{14(62.4)\left(\frac{4}{3}\right)\left[\frac{3}{2}(3)^2 - \frac{1}{3}(3)^3\right]}$$

10. $F(x) = K x^{-2}$ $\quad x =$ DISTANCE BETWEEN $e_1 \,\&\, e_2$

10 DYNES $= 10 \times 10^{-5}$ N $= K(2)^{-2}$ $\Rightarrow$ $4(10 \times 10^{-5})$ N$\cdot$cm $= K$

$$F(x) = 40 \times 10^{-5} x^{-2}$$

$$W = \int_1^5 40 \times 10^{-5} x^{-2}\, dx$$

$$= -40 \times 10^{-5} x^{-1} \Big|_1^5$$

$$= -40 \times 10^{-5}\left[\tfrac{1}{5} - 1\right] = \boxed{\tfrac{4}{5}(40 \times 10^{-5}) \text{ N}\cdot\text{cm}}$$