HW 7
SOLUTIONS

1. a) $y = u^3 \quad u = x^2 + x + 1$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = (3u^2)(2x+1) = \boxed{3(x^2+x+1)^2(2x+1)}$$

b) $y = u^2 \quad u = \cos x$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = (2u)(-\sin x) = \boxed{-2\cos x \sin x}$$

c) $y = \sin\left(\frac{1}{x+1}\right)$

$$\boxed{\frac{dy}{dx} = \cos\left(\frac{1}{x+1}\right)\left(\frac{-1}{(x+1)^2}\right)}$$

d) $y = \sin^{57}(x^3 + 2x^2 + 1)$

$$\boxed{\frac{dy}{dx} = 57\sin^{56}(x^3+2x^2+1)\cos(x^3+2x^2+1)(3x^2+4x)}$$

e) $y = \tan^2(\cos x)$

$$\boxed{\frac{dy}{dx} = 2\tan(\cos x)\sec^2(\cos x)(-\sin x)}$$

f) $y = \cos\left[(x^2+1)^4\right]$

$$\boxed{\frac{dy}{dx} = -\sin\left[(x^2+1)^4\right]\left(4(x^2+1)^3\right)(2x)}$$

2. $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

For $x = 0$ $\quad \left.\frac{du}{dx}\right|_{x=0} \approx 4 \quad \left.\frac{dy}{du}\right|_{u=0} \approx 1 \quad$ So $\boxed{\left.\frac{dy}{dx}\right|_{x=0} \approx 4}$

For $x = 1$ $\quad \left.\frac{du}{dx}\right|_{x=1} \approx \frac{3}{2} \quad \left.\frac{dy}{du}\right|_{u=3} \approx \frac{1}{2} \quad$ So $\boxed{\left.\frac{dy}{dx}\right|_{x=1} \approx \frac{3}{4}}$

For $x = 2$, $\quad \left.\frac{du}{dx}\right|_{x=2} \approx 0, \quad \left.\frac{dy}{du}\right|_{u=4} \approx k \quad$ So $\boxed{\left.\frac{dy}{dx}\right|_{x=2} = 0}$

For $x = 3$, $\quad \left.\frac{du}{dx}\right|_{x=3} \approx -2, \quad \left.\frac{dy}{du}\right|_{u=3} \approx \frac{1}{2} \quad$ So $\boxed{\left.\frac{dy}{dx}\right|_{x=3} \approx -1}$

For $x = 4$, $\quad \left.\frac{du}{dx}\right| \approx -4 \quad \left.\frac{dy}{du}\right| \approx 1 \quad$ So $\left.\frac{dy}{\ }\right| \approx$

3. $r_0 = 10$ KM

RADIUS INCREASES AT A RATE OF 40,000KM/hr

$\frac{dV}{dt} = ?$ WHEN $r = 40{,}000$? $r = 1{,}000{,}000$ KM?

$$V = \frac{4}{3}\pi r^3 \qquad r = 10 + 40{,}000t$$

$$\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt} = (4\pi r^2)(40{,}000)$$

$$\left.\frac{dV}{dt}\right|_{r=40{,}000} = 4\pi(40{,}000)^2(40{,}000) = \boxed{2.56\pi \times 10^{14}\ \text{km}^3/\text{hr}}$$

$$\left.\frac{dV}{dt}\right|_{r=1{,}000{,}000} = 4\pi(1{,}000{,}000)^2(40{,}000) = \boxed{1.6\pi \times 10^{17}\ \text{km}^3/\text{hr}}$$

4. LET $f(x) = \begin{cases} x^2 \sin\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

a) FIND $f'(x)$ FOR $x \neq 0$ USING THE PRODUCT RULE AND CHAIN RULE:

$$f'(x) = 2x\sin\left(\tfrac{1}{x}\right) + x^2\cos\left(\tfrac{1}{x}\right)\left(-\tfrac{1}{x^2}\right)$$

$$\boxed{f'(x) = 2x\sin\left(\tfrac{1}{x}\right) - \cos\left(\tfrac{1}{x}\right)}$$

b) FIND $f'(x)$ FOR $x = 0$ USING THE DEFINITION

$$f'(x) = \lim_{h\to 0}\frac{f(0+h) - f(0)}{h} = \lim_{h\to 0}\frac{h^2\sin\left(\frac{1}{h}\right)}{h} = \lim_{h\to 0} h\sin\left(\tfrac{1}{h}\right)$$

& BY THE SQUEEZE THM, SINCE $-|h| \leq h\sin\left(\frac{1}{h}\right) \leq |h|$

AND $\lim_{h\to 0} -|h| = \lim_{h\to 0}|h| = 0$, SO $\boxed{f'(x) = 0}$ AT $x = 0$.

c) FOR $f'(x)$ TO BE CONTINUOUS WE MUST HAVE:

i) $0 \in f'(x)$ ✓

ii) $\lim_{x\to 0} f'(x)$ EXISTS

iii) $\lim_{x\to 0} f'(x) = f'(0)$

BUT $\lim_{x\to 0} f'(0) = \lim_{x\to 0} 2x\sin\left(\tfrac{1}{x}\right) - \cos\left(\tfrac{1}{x}\right)$

$= 0 -$ DNE

5. PROVE THE QUOTIENT RULE USING THE PRODUCT RULE AND CHAIN RULE.

i.e. PROVE IF $y = \frac{f(x)}{g(x)}$, THEN $y' = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$

PROOF:

$$y = \frac{f(x)}{g(x)} = f(x) \cdot g^{-1}(x)$$

$$\Rightarrow y' = f'(x)\, g^{-1}(x) + f(x) \frac{d}{dx}\left(g^{-1}(x)\right)$$

$$= f'(x)\, g^{-1}(x) + f(x)\left(-1\left(g^{-2}(x)\right) g'(x)\right)$$

$$= \left[f'(x)\, g^{-1}(x) - f(x)\, g^{-2}(x)\, g'(x)\right] \cdot \frac{g^2(x)}{g^2(x)}$$

$$= \frac{f'(x)\, g(x) - f(x)\, g'(x)}{g^2(x)}$$

6. a) $y = x^3 + 4x^4 + 3x + 5$

$$\frac{dy}{dx} = 3x^2 + 16x^3 + 3$$

$$\frac{d^2y}{dx^2} = 6x + 48x^2$$

$$\boxed{\frac{d^3y}{dx^3} = 6 + 96x}$$

b) $y = x^2 + 1$

$$\frac{dy}{dx} = 2x$$

$$\frac{d^2y}{dx^2} = 2$$

$$\boxed{\frac{d^3y}{dx^3} = 0}$$

c) $y = \cos(3x)$

$$\frac{dy}{dx} = -3\sin 3x$$

$$\frac{d^2y}{dx^2} = -9\cos 3x$$

$$\boxed{\frac{d^3y}{dx^3} = 27\sin 3x}$$

d) $y = \sin(x^2)$

$$\frac{dy}{dx} = \cos(x^2) \cdot 2x$$

$$\frac{d^2y}{dx^2} = -\sin(x^2)(2x)^2 + 2\cos(x^2)$$

$$= -4x^2\sin(x^2) + 2\cos(x^2)$$

$$\frac{d^3y}{dx^3} = -8x\sin(x^2) - 4x^2\cos(x^2)(2x) - 2\sin(x^2)(2x)$$

$$\boxed{= -12x\sin x^2 - 8x^3\cos(x^2)}$$

e) $y = \frac{1}{x^2+1} = (x^2+1)^{-1}$

$$\frac{dy}{dx} = -(x^2+1)^{-2}(2x) = -2x(x^2+1)^{-2}$$

$$\frac{d^2y}{dx^2} = -2(x^2+1)^{-2} + 4x(x^2+1)^{-3}(2x$$

$$= -2(x^2+1)^{-2} + 8x^2(x^2+1)^{-3}$$

$$\frac{d^3y}{dx^3} = 4(x^2+1)^{-3}(2x) + 16x(x^2+1)^{-3}$$

$$+ 8x^2(-3)(x^2+1)^{-4}(2x)$$

$$\boxed{= 20x(x^2+1)^{-3} - 48x^3(x^2+1)^{-4}}$$

7. $s = 3t^3 - 5t^2 + t + 1$

a) $\boxed{v(t) = s'(t) = 9t^2 - 10t + 1}$

$\boxed{a(t) = v'(t) = s''(t) = 18t - 10}$

b) S INCREASING WHEN $v > 0$

$\Rightarrow 9t^2 - 10t + 1 > 0$

$\Rightarrow (9t - 1)(t - 1) > 0$

$+ \quad - \quad +$; $\frac{1}{9}$; 1

So WHEN $\boxed{-\infty < t < \frac{1}{9} \quad \text{OR} \quad 1 < t < \infty}$

c) S DECREASING WHEN $v < 0$

$\Rightarrow \frac{1}{9} < t < 1$

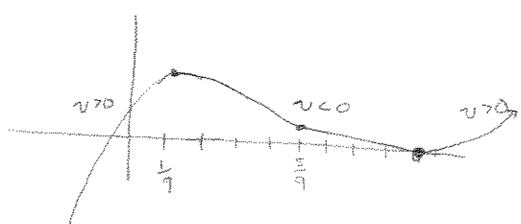
d) SLOWING DOWN WHEN $a < 0$

$\Rightarrow 18t - 10 < 0$

$18t < 10$

$\boxed{t < \frac{5}{9}}$

e)



8. $s = 20t - 5t^2$

a) $v(t) = s'(t) = 20 - 10t$

$\boxed{v(0) = 20 \text{ m/s}}$

b) $v(t) = 0$ WHEN $20 - 10t = 0$

$20 = 10t$

$t = 2\text{ s.}$

MAX HEIGHT AT $\boxed{t = 2s}$

c) $s(2) = 20(2) - 5(2)^2$

$= 40 - 20$

$\boxed{= 20m}$

d) $v(1) = 20 - 10 = \boxed{10 \text{ m/s}}$

$\Rightarrow$ S TRAVELING IN (+) DIRECTION

e) $0 = 20t - 5t^2 = 5t(4 - t) \Rightarrow$

HITS GROUND AT $\boxed{t = 4s}$

9. a) $x^2+y^2=1$

$2x+2y\frac{dy}{dx}=0$

$2y\frac{dy}{dx}=-2x$

$\boxed{\frac{dy}{dx}=\frac{-x}{y}}$

b) $xy=1$

$y+x\frac{dy}{dx}=0$

$x\frac{dy}{dx}=-y$

$\boxed{\frac{dy}{dx}=\frac{-y}{x}}$

c) $x^2y^2=x+1$

$2xy^2+2x^2y\frac{dy}{dx}=1$

$2x^2y\frac{dy}{dx}=1-2xy^2$

$\boxed{\frac{dy}{dx}=\frac{1-2xy^2}{2x^2y}}$

d) $\cos(xy)+\sin(xy)=\sqrt{2}$

$-\sin(xy)\left(y+x\frac{dy}{dx}\right)+\cos(xy)\left(y+x\frac{dy}{dx}\right)=0$

$x\cos(xy)\frac{dy}{dx}-x\sin(xy)\frac{dy}{dx}=y(\sin(xy)-\cos(xy))$

$\boxed{\frac{dy}{dx}=\frac{y(\sin(xy)-\cos(xy))}{x(\cos(xy)-\sin(xy))}}$

e) $x^2y+y^2x=0$

$2xy+x^2\frac{dy}{dx}+2y\frac{dy}{dx}x+y^2=0$

$\frac{dy}{dx}(x^2+2xy)=-y^2-2xy$

$\boxed{\frac{dy}{dx}=\frac{-y^2-2xy}{x^2+2xy}}$

10. a) $\left(-\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)$

$\frac{dy}{dx}=\frac{-\frac{\sqrt{2}}{2}}{-\frac{\sqrt{2}}{2}}=1$

$\boxed{y-\frac{\sqrt{2}}{2}=1\left(x+\frac{\sqrt{2}}{2}\right)}$

b) $\left(\frac{1}{4},4\right)$

$\frac{dy}{dx}=\frac{-4}{\frac{1}{4}}=-16$

$\boxed{y-4=-16\left(x-\frac{1}{4}\right)}$

c) $(1,\sqrt{2})$

$\frac{dy}{dx}=\frac{1-2(1)(\sqrt{2})^2}{2(1)^2(\sqrt{2})}=\frac{1-4}{2\sqrt{2}}=\frac{-3}{2\sqrt{2}}$

$\boxed{y-\sqrt{2}=\frac{-3}{2\sqrt{2}}(x-1)}$

d) $\left(\frac{\pi}{4},1\right)$

$\frac{dy}{dx}=\frac{1(\sin\frac{\pi}{4}-\cos\frac{\pi}{4})}{\frac{\pi}{4}(\cos\frac{\pi}{4}-\sin\frac{\pi}{4})}=\frac{0}{0}$

UNDEFINED $\Rightarrow$ VERTICAL LINE

$\boxed{x=\frac{\pi}{4}}$

e) $(-1,1)$

$\frac{dy}{dx}=\frac{-(1)^2-2(-1)(1)}{(-1)^2+2(-1)(1)}=\frac{-1+2}{1-2}=\frac{1}{-1}=-1$

$\boxed{y-1=-1(x+1)}$

11. a) $y = \sqrt{x} - x^{9/5}$

$$\boxed{\frac{dy}{dx} = \frac{1}{2}x^{-1/2} - \frac{9}{5}x^{4/5}}$$

b) $y = x^{-2/3}$

$$\boxed{\frac{dy}{dx} = \frac{-2}{3}x^{-5/3}}$$

c) $y = \frac{1}{\sqrt{x^2+1}} = (x^2+1)^{-1/2}$

$$\frac{dy}{dx} = \frac{-1}{2}(x^2+1)^{-3/2}(2x)$$

$$\boxed{= \frac{-x}{(\sqrt{x^2+1})^3}}$$

12. $x^2 + y^2 = 25$

$$2x + 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{-x}{y}$$

$$2 + 2\frac{dy}{dx}\cdot\frac{dy}{dx} + 2y\frac{d^2y}{dx^2} = 0$$

$$2 + 2\left(\frac{dy}{dx}\right)^2 + 2y\frac{d^2y}{dx^2} = 0$$

So $\left.\frac{dy}{dx}\right|_{(4,3)} = \frac{-4}{3} \Rightarrow \underline{2 + 2\left(-\frac{4}{3}\right)^2 + 2(3)\frac{d^2y}{dx^2} = 0}$

$$2 + 2\left(\frac{16}{9}\right) + 6\frac{d^2y}{dx^2} = 0$$

$$\frac{18+32}{9} + 6\frac{d^2y}{dx^2} = 0$$

$$6\frac{d^2y}{dx^2} = \frac{-50}{9}$$

$$\left.\frac{d^2y}{dx^2}\right|_{(4,3)} = \frac{-50}{6\cdot 9} = \boxed{\frac{-25}{27}}$$

13. $s^3t + s^2t^2 = 1$

$\frac{ds}{dt} = ?$

$$\left[s^3t + s^2t^2 = 1\right]\frac{d}{dt}$$

$$3s^2t\frac{ds}{dt} + s^3 + 2st^2\frac{ds}{dt} + 2s^2t = 0$$

$$\frac{ds}{dt}(3s^2t + 2st^2) = -s^3 - 2s^2t$$

$$\boxed{\frac{ds}{dt} = \frac{-s^3 - 2s^2t}{3s^2t + 2st^2}}$$

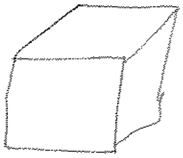
$\frac{dt}{ds} = ?$

$$\left[s^3t + s^2t^2 = 1\right]\frac{d}{ds}$$

$$3s^2t + s^3\frac{dt}{ds} + 2st^2 + 2s^2t\frac{dt}{ds} = 0$$

$$\frac{dt}{ds}(s^3 + 2s^2t) = -3s^2t - 2st^2$$

$$\boxed{\frac{dt}{ds} = \frac{-3s^2t - 2st^2}{s^3 + 2s^2t}}$$

14. 

$$\frac{ds}{dt} = 5\text{cm/s}$$

$$\frac{dV}{dt} = ? \text{ WHEN } s = 20\text{cm}?$$

$$V = s^3$$

$$\frac{dV}{dt} = 3s^2 \frac{ds}{dt}$$

$$\left.\frac{dV}{dt}\right|_{s=20} = 3(20)^2 \cdot 5 = 15(400) = \boxed{6000 \text{ cm}^3/\text{s}}$$

RATE OF SURFACE AREA : $\frac{dA_s}{dt}$

$$A_s = 6s^2$$

$$\frac{dA_s}{dt} = 12s \frac{ds}{dt}$$

$$\left.\frac{dA_s}{dt}\right|_{s=20} = 12(20) \cdot 5 = \boxed{1200 \text{ cm}^2/\text{s}}$$

15. $\frac{dr}{dt} = 1 \text{mm/s}$ $\quad \frac{dA}{dt} = ?$ WHEN $r = 10\text{cm} = 100\text{mm}$

$$A = \pi r^2$$

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

$$\left.\frac{dA}{dt}\right|_{r=10} = 2\pi(100)(1) = \boxed{20\cancel{0\pi} \text{ mm}^2/\text{s}}$$ ✓

$$2\pi(10)(0.1) = 6.28 \text{ cm}^2/\text{sec}$$ ✓

16. $R(t) = 10 + 40{,}000t - 10t^2$

a) $R_0 = 10 \text{ km}$

b) $\frac{dr}{dt} = 40{,}000 - 20t$

$$\boxed{\left.\frac{dr}{dt}\right|_{t=0} = 40{,}000 \text{ km/hr}}$$

c) $V = \frac{4}{3}\pi r^3$

$$V_0 = \frac{4}{3}\pi(10)^3$$

$$= \frac{4}{3}\pi(1000)$$

$$\boxed{= \frac{4000\pi}{3} \text{ km}^3}$$

d) $V = \frac{4}{3}\pi r^3$

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$$\left.\frac{dV}{dt}\right|_{t=0} = 4\pi(10)^2(40{,}000)$$

$$\boxed{= 1.6\pi \times 10^6 \text{ km}^3/\text{s}}$$

e) $\left.\frac{dr}{dt}\right|_{t=1} = 40{,}000 - 20$

$$\boxed{= 39980 \text{ km/s}}$$

f) $\left.\frac{dV}{dt}\right|_{t=1} = 4\pi(10 + 40{,}000 - 10)^2(3998$

g) $\left.\frac{dr}{dt}\right|_{t=24} = 40{,}000 - 20(24)$

h) $\left.\frac{dV}{dt}\right|_{t=24} = 4\pi r^2 \frac{dr}{dt}$

$= 4\pi\left(10 + 40{,}000(24) - 10(24)^2\right)\left(40{,}000 - 20(24)\right)$

i) $V(24) = \frac{4}{3}\pi r^3$

$= \frac{4}{3}\pi\left(10 + 40{,}000(24) - 10(24)^2\right)^3$