HW #6
SOLUTIONS

→ graded

1. MANY POSSIBLE SOLUTIONS

FOR EXAMPLE

$L(f(x)) = 4f(x).$

IT HAS A FUNCTION FOR THE INPUT + OUTPUT, SO IT IS A

OPERATOR AND,

a) $L(f(x)+g(x)) = [f(x)+g(x)]4 = 4f(x) + 4g(x) = L(f(x)) + L(g(x)).$

b) $L(kf(x)) = 4kf(x) = k[4f(x)] = kL(f(x))$

$\therefore$ $L(x) = 4f(x)$ IS A LINEAR OPERATOR.

2. a) $y = 3x^2$

$\boxed{y' = 6x}$

b) $y = kx$

$\boxed{y' = k}$

c) $y = x^9 + x^7 + x^5 + x^3 + x + 1$

$\boxed{y' = 9x^8 + 7x^6 + 5x^4 + 3x^2 + 1}$

d) $y = \frac{1}{x} + x^2 = x^{-1} + x^2$

$y' = \boxed{-x^{-2} + 2x}$

$= \boxed{\frac{-1}{x^2} + 2x}$

e) $y = \frac{\pi}{x^3} = \pi x^{-3}$

$y' = \boxed{-3\pi x^{-4} = \frac{-3\pi}{x^4}}$

f) $y = (x^2+x+1)(x-1) = x^3 - x^2 + x^2 - x + x - 1$

$y = x^3 - 1$

$\boxed{y' = 3x^2}$

g) $y = \frac{x^2+x+1}{x-1}$

$\boxed{y' = \frac{(2x+1)(x-1) + (1)(x^2+x+1)}{(x-1)^2}}$

h) $y = 17x^3\left(\frac{x+1}{x^2-x+1}\right)$

$y' = 17(3)x^2\left(\frac{x+1}{x^2-x+1}\right) + 17x^3\left(D_x\left(\frac{x+1}{x^2-x+1}\right)\right)$

$= \cancel{41}\,51x^2\left(\frac{x+1}{x^2-x+1}\right) + 17x^3\left(\frac{1(x^2-x+1) - (x+1)(2x-\ldots}{(x^2-x+1)^2}\right.$

i) $y = \frac{3}{23x^2} - \frac{13}{2} = \frac{3}{23}x^{-2} - \frac{13}{2}$

$\boxed{y' = \frac{-6}{23}x^{-3}}$

j) $y = 2x^{-5} + 3x^{-3} + x^{-1}$

$\boxed{y' = -10x^{-6} - 9x^{-4} - x^{-2}}$

k) $y = x^{987654321}$

$\boxed{y' = 987654321\,x^{987654320}}$

l) $\dot{y} = (x-4)^2 = (x-4)(x-4)$

$= x^2 - 8x + 16$

$\boxed{y' = 2x - 8}$

m) $y = (x^2+1)^3 = (x^2+1)(x^2+1)(x^2+1)$

$= (x^4 + 2x^2 + 1)(x^2+1)$

$= x^6 + 3x^4 + 3x^2 + 1$

$\boxed{y' = 6x^5 + 12x^3 + 6x}$

3. a) $D_x [f(x)]^2 = 2 \cdot f(x) \cdot D_x f(x)$

Pf: $D_x [f(x)]^2 = D_x [f(x) \cdot f(x)] = D_x f(x) \cdot f(x) + f(x) \cdot D_x f(x)$

$= 2 \cdot f(x) \cdot D_x f(x)$ ✓

b) $D_x \{f(x)\}^3 = 3\{f(x)\}^2 \cdot D_x f(x)$

Pf: $D_x [f(x)]^3 = D_x [f(x) \cdot (f(x) \cdot f(x))] = D_x f(x) \cdot f(x) \cdot f(x) + f(x) \cdot D_x [f(x)]^2$

$= D_x f(x) \cdot f^2(x) + 2 \cdot f^2(x) \cdot D_x f(x) = 3 f^2(x) \cdot D_x f(x)$ ✓

c) $D_x [f(x)]^n = n f^{n-1}(x) \cdot D_x [f(x)]$

4. $f(1) = 1, \quad f'(1) = 2, \quad g(1) = 3, \quad g'(1) = 4$

a) $(f+g)'(1) = f'(1) + g'(1) = 2 + 4 = \boxed{6}$

b) $(f \cdot g)'(1) = f'(1) \cdot g(1) + f(1) g'(1) = 2(3) + 1(4) = 6 + 4 = \boxed{10}$

c) $\left(\frac{f}{g}\right)'(1) = \frac{f'(1) \cdot g(1) - f(1) g'(1)}{g^2(1)} = \frac{2(3) - 1(4)}{3^2} = \frac{6-4}{3^2} = \boxed{\frac{2}{9}}$

5. Prove $(f-g)'(x) = f'(x) - g'(x)$

Pf: $(f-g)'(x) = (f + (-g))'(x) = f'(x) + (-g(x))' = f'(x) - g'(x)$ ✓

$y = \frac{1}{3}x^3 + \frac{1}{2}x^2 + 1$

$y' = x^2 + x$

TANGENT IS HORIZONTAL WHEN $y' = 0$

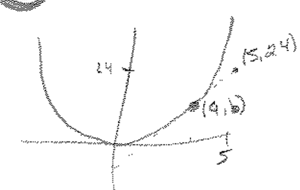
SO AT $0 = (x)(x+1)$

$x = 0$ & $x = -1$

CORRESPONDS TO THE POINTS

$\boxed{(0,1),\ (-1, \frac{7}{6})}$

$y = x^2$



$y' = 2x$ = SLOPE OF TANGENT LINE

EQUATION OF TANGENT LINE IS

$y - 24 = 2a(x - 5)$ & WE KNOW $b = a^2$

AND $b - 24 = 2a(a-5)$ SO 2 EQUATIONS W/ 2 UNKNOWNS.

SOLVE BY SUBSTITUTION:

$a^2 - 24 = 2a(a-5)$

$a^2 - 24 = 2a^2 - 10a$

$a^2 - 10a + 24 = 0$

$(a-6)(a-4) = 0$

$a = 6,\ a = 4$ BUT THE FALCON IS TRAVELING

FROM LEFT TO RIGHT $\Rightarrow a = 4,\ b = 16$, SO THE POINT IS $\boxed{(4,16)}$

8. $y = x^2$ $\quad y = -x^2 + \frac{3}{2}x + 1$

$y' = 2x$ $\quad y' = -2x + \frac{3}{2}$

WANT $m_1 = \frac{-1}{m_2}$

i.e. $-2x + \frac{3}{2} = \frac{-1}{2x}$

$\Rightarrow 4x^2 - 3x = 1$

$4x^2 - 3x - 1 = 0$

$x = \frac{3 \pm \sqrt{9+16}}{8}$

$x = \frac{3 \pm 5}{8} = 1, \frac{-1}{4}$

BUT $x > 0 \Rightarrow \boxed{x = 1}$

CHECK: $m_1 = -2(1) + \frac{3}{2} = \frac{-1}{2}$

$m_2 = 2(1) = 2$

$\Rightarrow$ LINES ARE PERPENDICULAR

AT $x = 1$

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9. a) $y = 3\sin x + 2\cos x$

$\boxed{y' = 3\cos x - 2\sin x}$

b) $y = \sin^3 x = \sin x \cdot \sin x \cdot \sin x$

$y' = \sin x\,(D_x \sin x \cdot \sin x) + \cos x \cdot \sin^2 x$

$= \sin x\,(\sin x \cdot \cos x + \sin x \cdot \cos x) + \cos x \cdot \sin^2 x$

$\boxed{= 3\sin^2 x \cos x}$

c) $y = \tan x + \sin x$

$\boxed{y' = \sec^2 x + \cos x}$

d) $y = \sin x \cdot \cos x$

$y' = \sin x\,(-\sin x) + \cos x \cdot \cos x$

$\boxed{= \cos^2 x - \sin^2 x}$

e) $y = \frac{\sin x + \cos x}{\sin x - \cos x}$

$y' = \frac{(\cos x - \sin x)(\sin x - \cos x) - (\sin x + \cos x)(\cos x + \sin x)}{(\sin x - \cos x)^2} = \boxed{\frac{-(\sin x - \cos x)^2 - (\sin x + \cos x)^2}{(\sin x - \cos x)^2}}$

f) $y = x\sin x$

$\boxed{y' = \sin x + x\cos x}$

g) $y = x^3 \cos x$

$\boxed{y' = 3x^2 \cos x - x^3 \sin x}$

h) $y = \dfrac{x\sin x + \cos x}{x^2+1}$

$y' = \dfrac{(\sin x + x\cos x - \sin x)(x^2+1) - 2x(x\sin x + \cos x)}{(x^2+1)^2}$

$\boxed{= \dfrac{x\cos x(x^2+1) - 2x(x\sin x + \cos x)}{(x^2+1)^2}}$

i) $y = \sin^2 x + \cos^2 x = 1$

$\boxed{y' = 0}$

j) $y = \cos x - x$

$\boxed{y' = -\sin x - 1}$

10) $D_x \cos 3x = \lim\limits_{h \to 0} \dfrac{\cos(3(x+h)) - \cos 3x}{h} = \lim\limits_{h\to 0} \dfrac{\cos(3x+3h) - \cos 3x}{h}$

$= \lim\limits_{h\to 0} \dfrac{\cos 3x \cos 3h - \sin 3x \sin 3h - \cos 3x}{h}$

$= \lim\limits_{h\to 0} \cos 3x\left(\dfrac{\cos 3h - 1}{h}\right) - \sin 3x\left(\dfrac{\sin 3h}{h}\right)$

$= \lim\limits_{h\to 0} -3\cos 3x\left(\dfrac{1-\cos 3h}{3h}\right) - 3\sin 3x\left(\dfrac{\sin 3h}{3h}\right)$

$= -3\cos(3x)(0) - 3(\sin 3x)(1) = \boxed{-3\sin 3x}$

11. $y = \cos x$ AT $x = \frac{\pi}{2}$

SLOPE: $y' = -\sin x$

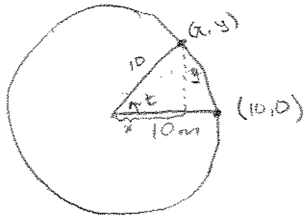
$y' = -\sin\left(\frac{\pi}{2}\right) = -1$

POINT $\left(\frac{\pi}{2}, \cos\frac{\pi}{2}\right) = \left(\frac{\pi}{2}, 0\right)$

EQUATION OF TANGENT LINE: $y - 0 = -1\left(x - \frac{\pi}{2}\right)$

$\boxed{y = -x + \frac{\pi}{2}}$

12.



ROTATES AT 1 RAD/SEC ⇒
AFTER t SECONDS
IT GOES THROUGH
t RADIANS

a) $\sin t = \frac{y}{10} \Rightarrow y = 10\sin t$

$\cos t = \frac{x}{10} \Rightarrow x = 10\cos t$

⇒ AT TIME t, THE SEAT IS AT $\boxed{(10\cos t, 10\sin t)}$.

b) $f(t) = 10\sin t$ ← VERTICAL

$\boxed{f'(t) = 10\cos t}$ ← HOW FAST IT'S RISING

c) $f'(t)$ HAS ITS LARGEST VALUE AT $t = 0$ i.e. WHEN $\cos t = 1$

SO IT IS RISING MOST QUICKLY AT $\boxed{t=0}$ W/ SPEED $f'(0) = \boxed{10\ldots}$

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13. a) $f(x) = (x+1)^{99}$

$\boxed{f'(x) = 99(x+1)^{98}}$

b) $f(x) = \sin(x^2)$

$\boxed{f'(x) = \cos(x^2)(2x)}$

c) $\omega = (t^2+1)^{100}$

$\boxed{\omega' = 100(t^2+1)^{99}(2t)}$

d) $\omega = (t^3+1)^{100}$

$\boxed{\omega' = 100(t^3+1)^{99}(3t^2)}$

e) $f(x) = x\sin\left(\frac{1}{x}\right)$

$f'(x) = \sin\left(\frac{1}{x}\right) + x\cos\left(\frac{1}{x}\right)(-x^{-2})$

$\boxed{= \sin\left(\frac{1}{x}\right) - \frac{1}{x}\cos\left(\frac{1}{x}\right)}$

f) $g(t) = \tan(\cos t)$

$\boxed{g'(t) = \sec^2(\cos t)(-\sin \ldots}$

g) $f(x) = \sin^2 x$

$\boxed{f'(x) = 2\sin x\cos x}$

h) $h(t) = \frac{1}{\sin^2 t} = \sin^{-2} t$

$\boxed{h'(t) = -2\sin^{-3} t\cos t}$

i) $y = \sin^3(x^2+x+1)$

$\boxed{y' = 3\sin^2(x^2+x+1)(\cos(x^2+x+1))(2x+1)}$

j) $z = \cos^n x$

$\boxed{z' = n\cos^{n-1} x(-\sin x)}$

k) $\omega = x^2\sin(x^2)$

$\boxed{\omega' = 2x\sin(x^2) + x^2\cos(x^2)(2x)}$

14. $f(2)=1$ $g(4)=2$

$f(4)=3$ $g(3)=4$

$f'(2)=5$ $g'(4)=6$

$f'(4)=7$ $g'(3)=8$

a) $h(4)$ IF $h(x)=f(g(x))$

$$h(4)=f(g(4))=f(2)=\boxed{1}$$

b) $h'(4)$ IF $h(x)=f(g(x))$

$$h'(x)=f'(g(x))\cdot g'(x)$$

$$h'(4)=f'(g(4))\cdot g'(4)=f'(2)\cdot 6=5\cdot 6=\boxed{30}$$

c) $h(4)$ IF $h(x)=g(f(x))$

$$h(4)=g(f(4))=g(3)=\boxed{4}$$

d) $h'(4)$ IF $h(x)=g(f(x))$

$$h'(x)=g'(f(x))\cdot f'(x)$$

$$h'(4)=g'(f(4))\cdot f'(4)=g'(3)\cdot 7=8\cdot 7=\boxed{56}$$

e) $h'(4)$ IF $h(x)=\frac{f(x)}{g(x)}$

$$h'(x)=\frac{f'(x)\cdot g(x)-g'(x)\cdot f(x)}{g^2(x)}$$

$$h'(4)=\frac{f'(4)\cdot g(4)-g'(4)\cdot f(4)}{g^2(4)}=\frac{7\cdot 2-6\cdot 3}{2^2}=\frac{14-18}{4}=\boxed{-1}$$

15. RADIUS RATE IS 10 cm/s. AFTER t SECONDS $r=10t$ cm

SO $A=\pi r^2 \Rightarrow \boxed{A(t)=\pi(10t)^2=100\pi t^2}$

$A'(t)=200\pi t$ SO AT ~~r=20cm,~~ $t=2$

AND $A'(2)=200\pi(2)=\boxed{400\pi\ cm^2/s}$ = RATE OF AREA WHEN $r=20$cm

~~RATE OF AREA WHEN R=20cm.~~

16. LET f BE AN ODD FUNCTION. i.e, $f(-x) = -f(x)$,

THEN CONSIDER $f(-x) = h(g(x))$ WHERE $g(x) = -x$ AND $h(x) = f(x)$.

THEN $D_x(f(-x)) = D_x(h(g(x))) = D_x\, h(g(x)) \cdot D_x\, g(x) = D_x\,(f(-x)) \cdot D_x(-x)$

$= D_x(-f(x))(-1) = -1(D_x(f(x))) \cdot (-1) = D_x(f(x))$

SO WHEN f IS ODD, $D_x f(x)$ IS EVEN.

LET f BE AN EVEN FUNCTION, i.e. $f(-x) = f(x)$.

THEN CONSIDER, AS ABOVE, $f(-x) = h(g(x))$ WHERE $h(x) = f(x)$, $g(x) = -x$.

THEN $D_x(f(-x)) = D_x(h(g(x))) = D_x(h(g(x))) \cdot D_x(g(x))$

$= D_x(f(-x)) \cdot D_x(-x) = D_x(f(x)) \cdot (-1) = -D_x(f(x))$

SO WHEN f IS EVEN, $D_x f(x)$ IS ODD.