Math 1210-1 Homework 4

Solutions

Show all work. Please box your answers. Be sure to write in complete sentences when a ppropriate. Also, I prefer exact answers like $\sqrt{2}$ instead of 1.414. Note that a symbol $\boxplus$ indicates that graph paper might be useful for that problem.

Limits Involving Trigonometric Functions

1. Evaluate each of the following limits.

   (a) $\lim_{t\to 0} \frac{\sin t}{1+t} = \frac{\sin 0}{1+0} = \frac{0}{1} = 0.$

   (b) $\lim_{t\to 0} t\sin t = 0\cdot \sin 0 = 0\cdot 0 = 0.$

   (c) $\lim_{t\to 0} \frac{\sin^2 t}{t} = \lim \sin t \frac{\sin t}{t} = (\sin 0)\cdot 1 = 0\cdot 1 = 0$

   (d) $\lim_{t\to 0} \frac{1-\cos^2 t}{t} = \lim \frac{\sin^2 t}{t} = 0$ by part c.

   (e) $\lim_{t\to 0} \frac{\sin^2 t}{t(1+\cos t)} = \lim \frac{\sin t}{t}\cdot \frac{\sin t}{1+\cos t} = 1\cdot \frac{\sin 0}{1+\cos 0} = 1\cdot \frac{0}{1} = 1\cdot 0 = 0.$

Asymptotic Limits and Infinite Limits

2. Find the limits:

   (a) $\lim_{x\to\infty} \frac{x}{x+2} = \lim \frac{x}{x+2}\cdot \frac{\frac{1}{x}}{\frac{1}{x}} = \lim \frac{1}{1+\frac{2}{x}} = \frac{1}{1+0} = 1$

   (b) $\lim_{x\to\infty} \frac{x^2}{x^3+2} = \lim \frac{x^2}{x^3+2}\cdot \frac{\frac{1}{x^3}}{\frac{1}{x^3}} = \lim \frac{\frac{1}{x}}{1+\frac{2}{x^3}} = \frac{0}{1+0} = 0$

   (c) $\lim_{x\to\infty} \frac{x^3}{x^2+2x+1} = \lim \frac{1}{\frac{1}{x}+\frac{1}{x^2}+\frac{1}{x^3}} = \frac{1}{0} = \infty$

   (d) $\lim_{\theta\to\infty} \frac{\sin\theta}{\theta^2+1}$. We know $-1 \le \sin\theta \le 1$, so $\frac{-1}{\theta^2+1} \le \frac{\sin x}{\theta^2+1} \le \frac{1}{\theta^2+1}$, and $\lim_{\theta\to\infty} \frac{-1}{\theta^2+1} = \lim \frac{\frac{-1}{\theta^2}}{1+\frac{1}{\theta^2}} = \frac{0}{1} = 0$. Also, $\lim_{\theta\to\infty} \frac{1}{\theta^2+1} = \lim \frac{\frac{1}{\theta^2}}{1+\frac{1}{\theta^2}} = \frac{0}{1} = 0$, so by the Squeeze Theorem, $\lim_{\theta\to\infty} \frac{\sin\theta}{\theta^2+1} = 0.$

   (e) $\lim_{x\to -2^+} \frac{x}{x+2} = \frac{-2}{0^+} = -\infty$

   (f) $\lim_{x\to 0^+} \frac{|x|}{x}$. If x is approaching 0 from the positive side, then $|x| = x$, so $\lim_{x\to 0^+} \frac{|x|}{x} = \lim \frac{x}{x} = \lim 1 = 1.$

   (g) $\lim_{x\to 0^+} \frac{\sin x}{1-\cos x} = \lim \frac{\sin x}{1-\cos x}\cdot \frac{1+\cos x}{1+\cos x} = \lim \frac{\sin x(1+\cos x)}{1-\cos^2 x} = \lim \frac{\sin x(1+\cos x)}{\sin^2 x} = \lim \frac{1+\cos x}{\sin x} = \frac{2}{0^+} = \infty$

3. We have given meaning to $\lim_{x\to A} f(x)$ for $A = a, a^-, a^+, -\infty, +\infty$. Moreover, this limit may be L (a finite number), $-\infty, +\infty$, or may fail to exist in any sense. This means that there are twenty possibilities. Give examples of $f(x)$ for each of these possibilities (either by formula, or by graph).

   There are many possible solutions.

4. Einstein's Special Theory of Relativity says that the mass $m(v)$ of an object is related to its velocity v by

$$m(v) = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}},$$

where c is the speed of light in a vacuum, and m_0 is the rest mass of the object. What is $\lim_{v \to c^-} m(v)$?

$$\lim_{x \to c^-} m(v) = \lim_{x \to c^-} \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{m_0}{0^+} = \infty$$

Continuity of Functions

5. For each of the following functions, determine whether or not it is continuous at 2. If it is not continuous at 2, explain why not.

   (a) $f(x) = (x+1)(x-2)$. This function is continuous at 2 because $\lim_{x \to 2} f(x) = f(2)$.

   (b) $f(x) = \dfrac{x+1}{x-2}$. This function is discontinuous at 2 since the function is not defined at $x = 2$, i.e. 2 is not in the domain of f.

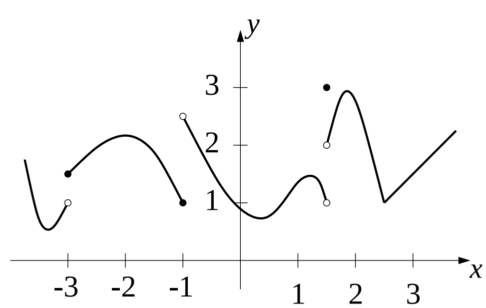
   (c) $f(x) = \dfrac{x^2-4}{x-2}$. This function is discontinuous at 2 since 2 is not in the domain of f.

   (d) $f(x) = \begin{cases} \dfrac{x^2-4}{x-2} & \text{if } x \neq 2 \\ 4 & \text{if } x = 2 \end{cases}$. This function is continuous at 2 since $\lim_{x \to 2} f(x) = f(2)$.

   (e) $f(x) = \begin{cases} x & \text{if } x > 2 \\ 2 & \text{if } x \leq 2 \end{cases}$. This function is continuous at 2 since $\lim_{x \to 2} f(x) = f(2)$.

   (f) $f(x) = [[x]]$ (This is the "greatest integer" function). This function is discontinuous at 2 since $\lim_{x \to 2} f(x)$ does not exist, so the limit $\neq f(2)$.

6. From the graph $y = f(x)$ below, state the intervals on which $f(x)$ is continuous.



f is continuous on $(-\infty, -3), [-3, -1], (-1, 1.5)$, and $(1.5, \infty)$

7. Each of the following functions are not defined somewhere. At what point are they not defined. We can explicitly define them at this point. What value should we assign so that the function is continuous at this point?

   (a) $f(x) = \dfrac{x^2-9}{x+3}$. $f(x)$ is not defined at $x = -3$. For $f(x)$ to be a continuous function we need $\lim_{x \to -3} f(x) = f(-3)$. $\lim_{x \to -3} f(x) = \lim \dfrac{(x+3)(x-3)}{x+3} = \lim x - 3 = -6$. So if we let $f(-3) = -6, f(x)$ will be continuous at $x = -3$.

   (b) $g(x) = \dfrac{1 - \cos x}{x}$. $g(x)$ is not defined at $x = 0$, and $\lim_{x \to 0} g(x) = 0$, so if we let $g(0) = 0, g(x)$ will be a continuous function.

(c) $h(x) = \dfrac{\sin x}{x}$. $h(x)$ is not defined at $x = 0$, and $\lim\limits_{x \to 0} h(x) = 1$, so if we let $h(0) = 1$, $h(x)$ will be a continuous function.

(d) $k(x) = x \sin\left(\frac{1}{x}\right)$. $k(x)$ is not defined for $x = 0$, so we need to find $\lim\limits_{x \to 0} k(x)$. Since we know $-1 \leq \sin(\frac{1}{x}) \leq 1$, then $-|x| \leq x \sin(\frac{1}{x}) \leq |x|$, and $\lim\limits_{x \to 0} -|x| = 0$, and $\lim\limits_{x \to 0} |x| = 0$, so by the Squeeze Theorem, $\lim\limits_{x \to 0} k(x) = 0$, so if we let $k(0) = 0$, $k(x)$ will be a continuous function.

(e) $m(x) = \dfrac{x^3}{x(2 + \cos x)}$. $m(x)$ is not defined at $x = 0$, and $\lim\limits_{x \to 0} m(x) = \lim \dfrac{x^2}{2 + \cos x} = \dfrac{0}{3} = 0$, so if we let $m(0) = 0$, $m(x)$ will be a continuous function.

8. ⊞ Suppose that f is a function which is continuous everywhere except at a (where $f(x) = c$). Furthermore, $\lim\limits_{x \to a} f(x) = b$ (and $b \neq c$). How can we get a functions that agrees with f everywhere except at a, and is continuous everywhere? Draw a picture.

   Let $f^*(x) = f(x)$ for all $x \neq a$, and $f^*(x) = b$ for $x = a$. Then $f^*(x) = f(x)$ for all $x \neq a$, and $\lim\limits_{x \to a} f^*(x) = \lim\limits_{x \to a} f(x) = b = f^*(a)$, so $f^*(x)$ is continuous everywhere.

9. ⊞ Sketch the graph of a function f which satisfies all of the following properties:

   (i) The domain of f is $[0, 5]$.

   (ii) $f(0) = f(2) = f(3) = f(4) = 1$.

   (iii) f is discontinuous at 2,3, and 4.

   (iv) f is right-continuous at 2, left-continuous at 4, and neither right- nor left-continuous at 3.

   There are many possible solutions.

10. ⊞ Let

$$f(x) = \begin{cases} |x| & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}.$$

   Sketch a graph of f as best as you can and find the single point where f is continuous. Why is f continuous at this point?

   f is continuous at 0 since $\lim\limits_{x \to 0} f(x) = 0 = f(0)$. But we need to prove the limit is 0. So given $\epsilon > 0$, there exists a $\delta > 0$ such that when $|x - 0| = |x| < \delta$, then $|x| < \epsilon$ since $y = |x|$ is a continuous function. But for this same δ, when $|x| < \delta$, then $|f(x)| < \epsilon$ since the irrational points in our δ neighborhood of x evaluate to 0 which is in our ϵ neighborhood of our supposed limit 0. Therefore, $\lim\limits_{x \to 0} f(x) = 0 = f(x)$, and f is continuous at 0.

11. Suppose that f is a function such that $\lim\limits_{x \to a} f(x) = b$. Prove that $\lim\limits_{x \to a} \sin(f(x)) = \sin b$.

   Proof: Let $g(x) = \sin x$. Let $\epsilon > 0$. Since $g(x)$ is continuous at all points, in particular, g is continuous at b, so there exists a $\delta_1 > 0$ such that when $|x - b| < \delta_1$, the $|\sin x - \sin b| < \epsilon$. But since $\lim\limits_{x \to a} f(x) = b$, there exists a $\delta_2 > 0$ such that when $|x - a| < \delta_2$, then $|f(x) - b| < \delta_1$ which implies when $|x - a| < \delta_2$ then $|\sin(f(x)) - \sin b| < \epsilon$. Therefore $\lim\limits_{x \to a} \sin(f(x)) = \sin(\lim\limits_{x \to a} f(x)) = \sin b$.

   Or we could say since $g(x) = \sin x$ is a continuous function, and since b is in the domain of g, by a theorem we did in class, the composition, $g \circ f = \sin(f(x))$ is continuous at a, so by the definition, $\lim\limits_{x \to a} \sin(f(x)) = \sin(\lim\limits_{x \to a} f(x)) = \sin b$.

12. Use the Intermediate Value Theorem to prove that $x^3 + 3x^2 - x - 3 = 0$ has a real solution between 0 and 2.
    Let $f(x) = x^3 + 3x^2 - x - 3$. Then $f(0) = -3$ and $f(2) = 15$, and since f is a continuous function, and 0 is between -3 and 15, by the Intermediate Value Theorem, there exists a c between 0 and 2 such that $f(c) = 0$, i.e. c is a solution to $x^3 + 3x^2 - x - 3 = 0$.

13. Let $f(x) = \dfrac{1}{x}$. Then $f(-1) = -1$ and $f(1) = 1$. Does the Intermediate Value Theorem imply that there is a point c in the interval $[-1, 1]$ such that $f(c) = 0$? Why or why not?
The Intermediate Value Theorem does not imply this because the function $f(x) = \frac{1}{x}$ is not continuous at 0, so we can not apply the theorem.

14. ⊞ Suppose that f is continuous on $[0, 1]$ and $0 \leq f(x) \leq 1$. Prove that f has a **fixed point** in $[0, 1]$. In other words, there exists c in $[0, 1]$ so that $f(c) = c$. Hint: Use the Intermediate Value Theorem and consider the function $g(x) = f(x) - x$.
Proof: Let $g(x) = f(x) - x$. Then since $0 \leq f(x) \leq 1$, we have $0 - x \leq g(x) \leq 1 - x \Longrightarrow -x \leq g(x) \leq 1$ when $x \in [0, 1]$. Since f is continuous, and $h(x) = x$ is continuous and the sum of two continuous functions is continuous, we have g is a continuous function. So by the Intermediate Value Theorem, there is a $c \in [0, 1]$ such that $g(c) = 0$ which means $f(c) - c = 0 \Longrightarrow f(c) = c$ and f has a fixed point in $[0, 1]$.