

Math 1210-1    Homework 4  
Due Tuesday February 3

Show all work. Please box your answers. Be sure to write in complete sentences when appropriate. Also, I prefer exact answers like  $\sqrt{2}$  instead of 1.414. Note that a symbol  $\boxplus$  indicates that graph paper might be useful for that problem.

### Limits Involving Trigonometric Functions

1. Evaluate each of the following limits.

(a)  $\lim_{t \rightarrow 0} \frac{\sin t}{1+t}$ .

(b)  $\lim_{t \rightarrow 0} t \sin t$ .

(c)  $\lim_{t \rightarrow 0} \frac{\sin^2 t}{t}$  (Hint: Use the Main Limit Theorem from last section together with a theorem from this section).

(d)  $\lim_{t \rightarrow 0} \frac{1 - \cos^2 t}{t}$ .

(e)  $\lim_{t \rightarrow 0} \frac{\sin^2 t}{t(1 + \cos t)}$  (Hint: do some algebraic manipulation first).

### Asymptotic Limits and Infinite Limits

2. Find the limits:

(a)  $\lim_{x \rightarrow \infty} \frac{x}{x+2}$

(b)  $\lim_{x \rightarrow \infty} \frac{x^2}{x^3+2}$

(c)  $\lim_{x \rightarrow \infty} \frac{x^3}{x^2+2x+1}$

(d)  $\lim_{\theta \rightarrow \infty} \frac{\sin \theta}{\theta^2+1}$

(e)  $\lim_{x \rightarrow -2^+} \frac{x}{x+2}$

(f)  $\lim_{x \rightarrow 0^+} \frac{|x|}{x}$

(g)  $\lim_{x \rightarrow 0^+} \frac{\sin x}{1 - \cos x}$

3. We have given meaning to  $\lim_{x \rightarrow A} f(x)$  for  $A = a, a^-, a^+, -\infty, +\infty$ . Moreover, this limit may be  $L$  (a finite number),  $-\infty, +\infty$ , or may fail to exist in any sense. This means that there are twenty possibilities. Give examples of  $f(x)$  for each of these possibilities (either by formula, or by graph).

4. Einstein's Special Theory of Relativity says that the mass  $m(v)$  of an object is related to its velocity  $v$  by

$$m(v) = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}},$$

where  $c$  is the speed of light in a vacuum, and  $m_0$  is the rest mass of the object. What is  $\lim_{v \rightarrow c^-} m(v)$ ?

## Continuity of Functions

5. For each of the following functions, determine whether or not it is continuous at 2. If it is not continuous at 2, explain why not.

(a)  $f(x) = (x + 1)(x - 2)$

(b)  $f(x) = \frac{x + 1}{x - 2}$

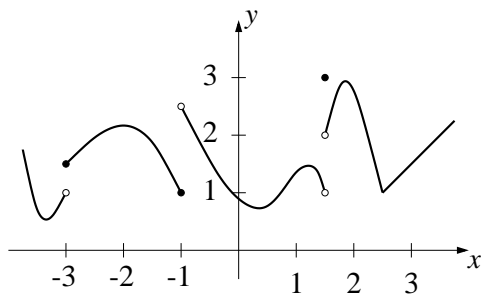
(c)  $f(x) = \frac{x^2 - 4}{x - 2}$

(d)  $f(x) = \begin{cases} \frac{x^2 - 4}{x - 2} & \text{if } x \neq 2 \\ 4 & \text{if } x = 2 \end{cases}$

(e)  $f(x) = \begin{cases} x & \text{if } x > 2 \\ 2 & \text{if } x \leq 2 \end{cases}$

(f)  $f(x) = \lceil x \rceil$  (This is the “greatest integer” function).

6. From the graph  $y = f(x)$  below, state the intervals on which  $f(x)$  is continuous.



7. Each of the following functions are not defined somewhere. At what point are they not defined. We can explicitly define them at this point. What value should we assign so that the function is continuous at this point?

(a)  $f(x) = \frac{x^2 - 9}{x + 3}$ .

(b)  $g(x) = \frac{1 - \cos x}{x}$ .

(c)  $h(x) = \frac{\sin x}{x}$ .

(d)  $k(x) = x \sin\left(\frac{1}{x}\right)$ .

(e)  $m(x) = \frac{x^3}{x(2 + \cos x)}$ .

8.  $\boxplus$  Suppose that  $f$  is a function which is continuous everywhere except at  $a$  (where  $f(x) = c$ ). Furthermore,  $\lim_{x \rightarrow a} f(x) = b$  (and  $b \neq c$ ). How can we get a functions that agrees with  $f$  everywhere except at  $a$ , and is continuous everywhere? Draw a picture.

9.  $\boxplus$  Sketch the graph of a function  $f$  which satisfies all of the following properties:

(i) The domain of  $f$  is  $[0, 5]$ .

(ii)  $f(0) = f(2) = f(3) = f(4) = 1$ .

(iii)  $f$  is discontinuous at 2, 3, and 4.

(iv)  $f$  is right-continuous at 2, left-continuous at 4, and neither right- nor left-continuous at 3.

10.  $\boxplus$  Let

$$f(x) = \begin{cases} |x| & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}.$$

Sketch a graph of  $f$  as best as you can and find the single point where  $f$  is continuous. Why is  $f$  continuous at this point?

11. Suppose that  $f$  is a function such that  $\lim_{x \rightarrow a} f(x) = b$ . Prove that  $\lim_{x \rightarrow a} \sin(f(x)) = \sin b$ .

12. Use the Intermediate Value Theorem to prove that  $x^3 + 3x^2 - x - 3 = 0$  has a real solution between 0 and 2.

13. Let  $f(x) = \frac{1}{x}$ . Then  $f(-1) = -1$  and  $f(1) = 1$ . Does the Intermediate Value Theorem imply that there is a point  $c$  in the interval  $[-1, 1]$  such that  $f(c) = 0$ ? Why or why not?

14.  $\boxplus$  Suppose that  $f$  is continuous on  $[0, 1]$  and  $0 \leq f(x) \leq 1$ . Prove that  $f$  has a **fixed point** in  $[0, 1]$ . In other words, there exists  $c$  in  $[0, 1]$  so that  $f(c) = c$ . Hint: Use the Intermediate Value Theorem and consider the function  $g(x) = f(x) - x$ .