

Math 1210-1 Homework 3 Solutions

Show all work. Please box your answers. Be sure to write in complete sentences when appropriate. Also, I prefer exact answers like $\sqrt{2}$ instead of 1.414. Note that a symbol \boxplus indicates that graph paper might be useful for that problem.

Introduction to Limits

1. Find the indicated limits.

(a) $\lim_{x \rightarrow 2} (x - 3) = 2 - 3 = -1$

(b) $\lim_{x \rightarrow -1} (x^2 + 3x - 4) = (-1)^2 + 3(-1) - 4 = -6$

(c) $\lim_{x \rightarrow -3} \frac{x^2 - 4x - 21}{x + 3} = \lim_{x \rightarrow -3} \frac{(x - 7)(x + 3)}{x + 3} = \lim_{x \rightarrow -3} x - 7 = -3 - 7 = -10$

(d) $\lim_{y \rightarrow 0} \frac{y^5 + 2y^4 + y^3}{y^3} = \lim_{y \rightarrow 0} \frac{y^3(y^2 + 2y + 1)}{y^3} = \lim_{y \rightarrow 0} y^2 + 2y + 1 = 1$

(e) $\lim_{t \rightarrow -5} \frac{\sqrt{(t+5)^3}}{t+5} = \lim_{t \rightarrow -5} \frac{(t+5)\sqrt{t+5}}{t+5} = \lim_{t \rightarrow -5} \sqrt{t+5} = 0$

2. Below are two important limits. Use a calculator to estimate the limit by taking smaller and smaller values for x . Be sure to check both sides of 0.

(a) $\lim_{x \rightarrow 0} \frac{\sin x}{x}$ Make a table of values for x and $y = \frac{\sin x}{x}$, and notice that the values look like they are going toward 1 as x gets close to 0.

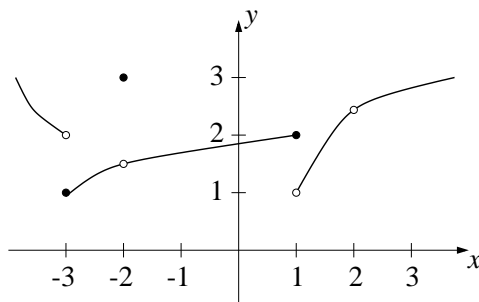
(b) $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x}$ Make a table of values for x and $y = \frac{1 - \cos x}{x}$ and notice that the values get close to 0 as x gets close to 0.

3. \boxplus Plot each of the following graphs on a small window near the origin. On each also plot $y = x$. Use these graphs to explain the results from Problem 2.

(a) $y = \sin x$ The graph of $\sin x$ looks a lot like the graph of $y = x$ as x is really close to 0, so $\lim_{x \rightarrow 0} \frac{\sin x}{x} \approx \lim_{x \rightarrow 0} \frac{x}{x} = \lim_{x \rightarrow 0} 1 = 1$.

(b) $y = 1 - \cos x$ The graph of $1 - \cos x$ is almost the horizontal line $y = 0$ when x is really close to 0. So $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x} \approx \lim_{x \rightarrow 0} \frac{0}{x} = \lim_{x \rightarrow 0} 0 = 0$.

4. For the function f graphed in the figure below, find the indicated limit or function value, or state that it does not exist.



(a) $\lim_{x \rightarrow -2} f(x) \approx 1.5$

(b) $f(-2) = 3$

- (c) $\lim_{x \rightarrow -3} f(x)$ does not exist
- (d) $f(-3) = 1$
- (e) $\lim_{x \rightarrow 1} f(x)$ does not exist
- (f) $f(1) = 2$
- (g) $\lim_{x \rightarrow 2} f(x) \approx 2.5$
- (h) $f(2)$ is not defined
- (i) $\lim_{x \rightarrow 1^+} f(x) = 1$
- (j) $\lim_{x \rightarrow -3^-} f(x) = 2$

5. \boxplus Sketch, as best you can, the graph of a function f that satisfies all of the following properties:

- (a) Its domain is the interval $[0, 4]$.
- (b) $f(0) = f(1) = f(2) = f(3) = f(4) = 1$.
- (c) $\lim_{x \rightarrow 1} f(x) = 2$.
- (d) $\lim_{x \rightarrow 2} f(x) = 1$.
- (e) $\lim_{x \rightarrow 3^-} f(x) = 2$.
- (f) $\lim_{x \rightarrow 3^+} f(x) = 1$.

There are many possible solutions.

6. Suppose that you carefully graphed the function $f(x) = x^3$. Then, during the night, a prankster mathematician changed the values of f at a million different places on your graph. Did this affect the value of $\lim_{x \rightarrow a} f(x)$ at any a ? Explain.

The missing points in the graph do not effect the limit at any a . The following is a rigorous proof of this statement. Let $f(x) = x^3$ and $f^*(x)$ be the function we are left with after the prankster changes the millions points. So given any $\epsilon > 0$ there exists a $\delta > 0$ such that when $|x - a| < \delta$ then $|f(x) - L| < \epsilon$. But it is not necessarily true that if $|x - a| < \delta$ then $|f^*(x) - L| < \epsilon$. If this δ does work, we are done. If not then since there are only a finite number of points which have been changed, there is a $\delta_1 \leq \delta$ such that δ_1 is the minimum distance from a to any of the removed points. This δ_1 now satisfies that when $|x - a| < \delta$ then $|f^*(x) - L| < \epsilon$ which means $\lim_{x \rightarrow a} f^*(x) = L$, and the limit value has not changed.

Rigorous Study of Limits

7. \boxplus For each of the following limits, sketch a possible graph for the function f and give the appropriate ϵ - δ definition.

There are many possibilities for the graphs.

- (a) $\lim_{x \rightarrow 1} f(x) = 3$. For all $\epsilon > 0$ there exists a $\delta > 0$ such that if $|x - 1| < \delta$ then $|f(x) - 3| < \epsilon$.
- (b) $\lim_{x \rightarrow a} f(x) = L$. For all $\epsilon > 0$ there exists a $\delta > 0$ such that if $|x - a| < \delta$ then $|f(x) - L| < \epsilon$.
- (c) $\lim_{x \rightarrow 2^+} f(x) = 1$, $\lim_{x \rightarrow 2^-} f(x) = 3$. For all $\epsilon_1 > 0$ there exists a $\delta_1 > 0$ such that if $0 < x - 2 < \delta_1$ then $|f(x) - 1| < \epsilon_1$, and for all $\epsilon_2 > 0$ there exists a $\delta_2 > 0$ such that if $0 < 2 - x < \delta_2$ then $|f(x) - 3| < \epsilon_2$.

- (d) $\lim_{x \rightarrow 0^-} f(x) = 0$, $\lim_{x \rightarrow 0^+}$ does not exist. For all $\epsilon_1 > 0$ there exists a $\delta_1 > 0$ such that when $0 < -x < \delta_1$ then $|f(x)| < \epsilon_1$, but there exists a $\epsilon_2 > 0$ such that for every $\delta_2 > 0$, if $0 < x < \delta_2$ then $|f(x)| > \epsilon_2$.
- (e) $\lim_{x \rightarrow c^+} f(x) = M$. For all $\epsilon > 0$ there exists a $\delta > 0$ such that if $0 < x - c < \delta$ then $|f(x) - M| < \epsilon$.
8. \boxplus Let f and g be functions such that $0 \leq f(x) \leq g(x)$ for all x near c , but possibly not at c . Prove that if $\lim_{x \rightarrow c} g(x) = 0$ then $\lim_{x \rightarrow c} f(x) = 0$. Draw a picture. This is a special case of the Squeeze Theorem from the next section, but you should prove this without using that theorem.
- Since $\lim_{x \rightarrow c} g(x) = 0$, we know for all $\epsilon > 0$ there exists a $\delta > 0$ such that if $|x - c| < \delta$ then $|g(x)| < \epsilon$, but $f(x) \leq g(x)$ which implies that $|f(x)| \leq |g(x)| < \epsilon$, so $|f(x)| < \epsilon$, or written in a more familiar way, $|f(x) - 0| < \epsilon$ which means $\lim_{x \rightarrow c} f(x) = 0$.

Limit Theorems

9. Evaluate each of the following limits using the limit theorems. State which theorem you use at each stage.
- (a) $\lim_{x \rightarrow 1} (3x + 2) = (by4.) \lim_{x \rightarrow 1} 3x + \lim_{x \rightarrow 1} 2 = (by3) 3 \lim_{x \rightarrow 1} x + \lim_{x \rightarrow 1} 2 = (by1and2) 3(1) + 2 = 5$.
- (b) $\lim_{x \rightarrow 0} (2x + 5)(x + 1) = (by6) \lim_{x \rightarrow 0} (2x + 5) \cdot \lim_{x \rightarrow 0} (x + 1) = (by4) (\lim_{x \rightarrow 0} 2x + \lim_{x \rightarrow 0} 5) (\lim_{x \rightarrow 0} x + \lim_{x \rightarrow 0} 1) = (by3) (2 \lim_{x \rightarrow 0} x + \lim_{x \rightarrow 0} 5) (\lim_{x \rightarrow 0} x + \lim_{x \rightarrow 0} 1) = (by1and2) (2(0) + 5)(0 + 1) = 5$.
- (c) $\lim_{x \rightarrow 2} \frac{x + 1}{x - 1} = (by7) \frac{\lim_{x \rightarrow 2} x + 1}{\lim_{x \rightarrow 2} x - 1} = (by4and5) \frac{\lim_{x \rightarrow 2} x + \lim_{x \rightarrow 2} 1}{\lim_{x \rightarrow 2} x - \lim_{x \rightarrow 2} 1} = (by1and2) \frac{2 + 1}{2 - 1} = 3$.
- (d) $\lim_{x \rightarrow -1} \sqrt{x^2 + 3} = (by9) \sqrt{\lim_{x \rightarrow -1} (x^2 + 3)} = (by4) \sqrt{\lim_{x \rightarrow -1} x^2 + \lim_{x \rightarrow -1} 3} = (by8, 1, and2) \sqrt{(-1)^2 + 3} = 2$.
- (e) $\lim_{t \rightarrow 2} \left(\frac{7t + 3}{t^2 + t + 1} \right)^{1/5} = (by8) \left(\lim_{t \rightarrow 2} \frac{7t + 3}{t^2 + t + 1} \right)^{1/5} = (by7) \left(\frac{\lim_{t \rightarrow 2} (7t + 3)}{\lim_{t \rightarrow 2} (t^2 + t + 1)} \right)^{1/5} = (by4) \left(\frac{\lim_{t \rightarrow 2} 7t + \lim_{t \rightarrow 2} 3}{\lim_{t \rightarrow 2} t^2 + \lim_{t \rightarrow 2} t + \lim_{t \rightarrow 2} 1} \right)^{1/5}$
 $(by1, 2, 3, and8) \left(\frac{7(2) + 3}{2^2 + 2 + 1} \right)^{1/5} = \left(\frac{17}{7} \right)^{1/5}$.
10. Suppose f and g are positive functions and $\lim_{x \rightarrow a} f(x) = 1$ and $\lim_{x \rightarrow a} g(x) = 2$. Find the following limits.
- (a) $\lim_{x \rightarrow a} f(x) - g(x) = 1 - 2 = -1$.
- (b) $\lim_{x \rightarrow a} f^2(x) + g^2(x) = 1^2 + 2^2 = 5$.
- (c) $\lim_{x \rightarrow a} \frac{f(x) - g(x)}{f(x) + g(x)} = \frac{1 - 2}{1 + 2} = \frac{-1}{3}$.
- (d) $\lim_{x \rightarrow a} (g^2(x) - f^2(x))^4 = (2^2 - 1^2)^4 = (4 - 1)^4 = 3^4 = 81$.
11. Find two functions f and g so that $\lim_{x \rightarrow 0} [f(x) + g(x)]$ exists but at least one of $\lim_{x \rightarrow 0} f(x)$ or $\lim_{x \rightarrow 0} g(x)$ does not exist.

There are many possible solutions, and I will only post two of them here.

Solution 1:

Let $f(x) = 0$ when $x < 0$ and $f(x) = 1$ when $x \geq 0$ and let $g(x) = 0$ when $x < 0$ and $g(x) = -1$ when $x \geq 0$. Then $\lim_{x \rightarrow 0} f(x)$ and $\lim_{x \rightarrow 0} g(x)$ do not exist, but $\lim_{x \rightarrow 0} [f(x) + g(x)] = 0$.

Solution 2:

Let $f(x) = \frac{1}{x}$. We need a function so that when we add it to $\frac{1}{x}$ we don't have x in the denominator.

So let $g(x) = -\frac{1}{x}$. Then for all x values not equal to 0, $f(x) + g(x) = 0$, so $\lim_{x \rightarrow 0} [f(x) + g(x)] = 0$, but $\lim_{x \rightarrow 0} f(x)$ and $\lim_{x \rightarrow 0} g(x)$ do not exist.

12. Find two functions f and g so that $\lim_{x \rightarrow 0} [f(x) \cdot g(x)]$ exists but at least one of $\lim_{x \rightarrow 0} f(x)$ or $\lim_{x \rightarrow 0} g(x)$ does not exist.

We know that for a function to not have a limit, there must either be a break in the graph at that x value, or we need to divide by 0. So let $f(x) = \frac{1}{x}$. We need to multiply this function by something so we are no longer dividing by x , so let $g(x) = x$. Then $\lim_{x \rightarrow 0} f(x)$ does not exist, but $\lim_{x \rightarrow 0} [f(x) \cdot g(x)] =$

$$\lim_{x \rightarrow 0} \frac{1}{x} \cdot x = \lim_{x \rightarrow 0} 1 = 1.$$

13. \boxplus Use the Squeeze Theorem to show that $\lim_{x \rightarrow 0} x \sin\left(\frac{1}{x}\right) = 0$. Draw a picture. Hint: Consider $f(x) = |x|$ and $g(x) = -|x|$.

Since $-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$ for any x , when we multiply through by $|x|$ we get $-|x| \leq x \sin x \leq |x|$, and

$$\lim_{x \rightarrow 0} -|x| = \lim_{x \rightarrow 0} |x| = 0, \text{ so by the squeeze theorem, we know } \lim_{x \rightarrow 0} x \sin\left(\frac{1}{x}\right) = 0.$$