Math 1210-1 Homework 3

Due Wednesday January 28

Show all work. Please box your answers. Be sure to write in complete sentences when appropriate. Also, I prefer exact answers like $\sqrt{2}$ instead of 1.414. Note that a symbol ⊞ indicates that graph paper might be useful for that problem.

Introduction to Limits

1. Find the indicated limits.

   (a) $\lim_{x\to 2}(x-3)$

   (b) $\lim_{x\to -1}(x^2+3x-4)$

   (c) $\lim_{x\to -3}\dfrac{x^2-4x-21}{x+3}$

   (d) $\lim_{y\to 0}\dfrac{y^5+2y^4+y^3}{y^3}$

   (e) $\lim_{t\to -5}\dfrac{\sqrt{(t+5)^3}}{t+5}$

2. Below are two important limits. Use a calculator to estimate the limit by taking smaller and smaller values for x. Be sure to check both sides of 0.

   (a) $\lim_{x\to 0}\dfrac{\sin x}{x}$

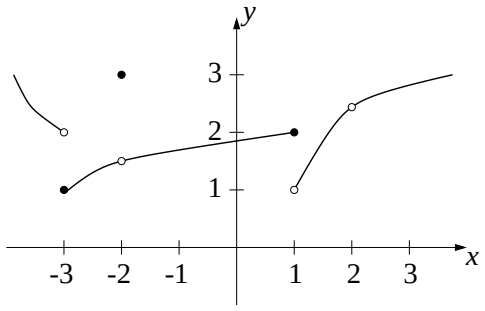
   (b) $\lim_{x\to 0}\dfrac{1-\cos x}{x}$

3. ⊞ Plot each of the following graphs on a small window near the origin. On each also plot $y = x$. Use these graphs to explain the results from Problem 2.

   (a) $y = \sin x$

   (b) $y = 1 - \cos x$

4. For the function f graphed in the figure below, find the indicated limit or function value, or state that it does not exist.



   (a) $\lim_{x\to -2} f(x)$

   (b) $f(-2)$

   (c) $\lim_{x\to -3} f(x)$

   (d) $f(-3)$

   (e) $\lim_{x\to 1} f(x)$

(f) $f(1)$

(g) $\lim_{x \to 2} f(x)$

(h) $f(2)$

(i) $\lim_{x \to 1^+} f(x)$

(j) $\lim_{x \to -3^-} f(x)$

5. ⊞ Sketch, as best you can, the graph of a function f that satisfies all of the following properties:

   (a) Its domain is the interval $[0, 4]$.

   (b) $f(0) = f(1) = f(2) = f(3) = f(4) = 1$.

   (c) $\lim_{x \to 1} f(x) = 2$.

   (d) $\lim_{x \to 2} f(x) = 1$.

   (e) $\lim_{x \to 3^-} f(x) = 2$.

   (f) $\lim_{x \to 3^+} f(x) = 1$.

6. Suppose that you carefully graphed the function $f(x) = x^3$. Then, during the night, a prankster mathematician changed the values of f at a million different places on your graph. Did this affect the value of $\lim_{x \to a} f(x)$ at any a? Explain.

Rigorous Study of Limits

7. ⊞ For each of the following limits, sketch a possible graph for the function f and give the appropriate ϵ-δ definition.

   (a) $\lim_{x \to 1} f(x) = 3$.

   (b) $\lim_{x \to a} f(x) = L$.

   (c) $\lim_{x \to 2^+} f(x) = 1, \quad \lim_{x \to 2^-} f(x) = 3$.

   (d) $\lim_{x \to 0^-} f(x) = 0, \quad \lim_{x \to 0^+}$ does not exist.

   (e) $\lim_{x \to c^+} f(x) = M$.

8. ⊞ Let f and g be functions such that $0 \leq f(x) \leq g(x)$ for all x near c, but possibly not at c. Prove that if $\lim_{x \to c} g(x) = 0$ then $\lim_{x \to c} f(x) = 0$. Draw a picture. This is a special case of the Squeeze Theorem from the next section, but you should prove this without using that theorem.

Limit Theorems

9. Evaluate each of the following limits using the limit theorems. State which theorem you use at each stage.

   (a) $\lim_{x\to 1}(3x+2)$.

   (b) $\lim_{x\to 0}(2x+5)(x+1)$.

   (c) $\lim_{x\to 2}\dfrac{x+1}{x-1}$.

   (d) $\lim_{x\to -1}\sqrt{x^2+3}$.

   (e) $\lim_{t\to 2}\left(\dfrac{7t+3}{t^2+t+1}\right)^{1/5}$.

10. Suppose f and g are positive functions and $\lim_{x\to a} f(x) = 1$ and $\lim_{x\to a} g(x) = 2$. Find the following limits.

    (a) $\lim_{x\to a} f(x) - g(x)$.

    (b) $\lim_{x\to a} f^2(x) + g^2(x)$.

    (c) $\lim_{x\to a}\dfrac{f(x)-g(x)}{f(x)+g(x)}$.

    (d) $\lim_{x\to a}\left(g^2(x) - f^2(x)\right)^4$.

11. Find two functions f and g so that $\lim_{x\to 0}[f(x)+g(x)]$ exists but at least one of $\lim_{x\to 0} f(x)$ or $\lim_{x\to 0} g(x)$ does not exist.

12. Find two functions f and g so that $\lim_{x\to 0}[f(x)\cdot g(x)]$ exists but at least one of $\lim_{x\to 0} f(x)$ or $\lim_{x\to 0} g(x)$ does not exist.

13. ⊞ Use the Squeeze Theorem to show that $\lim_{x\to 0} x\sin\left(\dfrac{1}{x}\right) = 0$. Draw a picture. Hint: Consider $f(x) = |x|$ and $g(x) = -|x|$.