MATH 6610
Fall 2016

Homework 5, Due December 2 2016

Show all the work. Please submit your working codes via e-mail. Late homework will not be accepted.

Problem 1.
a) Let A be a square matrix and let $||\cdot||$ be a consistent matrix norm (we say that $||\cdot||$ is compatible or consistent with a vector norm $||\cdot||$ if $||Ax|| \leq ||A||||x||$). Show that

$$\lim_{m\to\infty} ||A^m||^{1/m} = \rho(A)$$

b) *Bonus question for* 30 *points.*
Let A be a square matrix. Show that

$$\lim_{k\to\infty} A^k = 0 \text{ if and only if } \rho(A) < 1$$

Problem 2. (Computational Assignment).
Consider the 3×3 linear systems of the form $A_j x = b_j$, where b_j is always taken in such a way that the solution of the system is the vector $x = (1, 1, 1)^T$, and the matrices A_j are

a) $A_1 = \begin{pmatrix} 3 & 0 & 4 \\ 7 & 4 & 2 \\ -1 & 1 & 2 \end{pmatrix}$ b) $A_2 = \begin{pmatrix} -3 & 3 & -6 \\ -4 & 7 & -8 \\ 5 & 7 & -9 \end{pmatrix}$

c) $A_3 = \begin{pmatrix} 4 & 1 & 1 \\ 2 & -9 & 0 \\ 0 & -8 & -6 \end{pmatrix}$ d) $A_4 = \begin{pmatrix} 7 & 6 & 9 \\ 4 & 5 & -4 \\ -7 & -3 & 8 \end{pmatrix}$

e) Suggest strictly diagonally dominant by rows 3×3 matrix A_5
Implement Jacobi and Gauss-Seidel methods, test the methods on the examples a), b), c), d), e). Explain theoretically your numerical observations.
Problem 3.
1. (*Computational Part.*) Implement steepest descent algorithm. Test the method by suggesting two test problems. Discuss theoretically the results.
2.
a) Prove that for the conjugate gradient method, the residuals and search directions satisfy the relations

$$(r^k, r^j) = (p^k, Ap^j) = 0 \text{ for } k \neq j.$$

b) Use property a) to show that if matrix A is a $n \times n$ symmetric positive definite matrix, then the conjugate gradient algorithm converges in at most n steps.
c) (*Computational Part.*) Implement conjugate gradient (CG) algorithm. Test the method by suggesting two test problems and compare the performance of CG algorithm with the steepest descent algorithm. Discuss theoretically the results.