Problem 1. (Computational assignment. Please submit the codes by e-mail and include the printout and discussion of the results with the theoretical part.)
Implement the classical second order and the fourth order Runge-Kutta methods.
Solve $x - y' + 2yx = 0$ with $y(0) = 0$ on the interval $[0, 10]$ using the Runge-Kutta methods with $h = 0.1$.
Compare the approximate solution with the true solution $0.5(e^x - 1)$. Plot the graphs of the true and approximate solutions (for example, by plotting the logarithms of the solutions). Discuss the results.

Problem 2.
Derive a Taylor method of order 3 for the (IVP) below:
\[ y'(x) = -y(x) + 2\sin(x), \quad y(0) = -1 \]
What is the truncation error of the method? What is the exact solution for the IVP?

Problem 3.
Construct an example of (using definitions and theory discussed in class, such as root conditions, consistency condition, etc.):
a) a consistent but not stable linear multistep method
b) a stable but not consistent linear multistep method
What kind of behavior do you expect from the numerical solution produced by the methods in a) and in b)?

Problem 4.
Find the range of $a \in \mathbb{R}$ for which the method
\[ y_{n+2} + (a - 1)y_{n+1} - ay_n = \frac{h}{4} \left( (a + 3)f(t_{n+2}, y_{n+2}) + (3a + 1)f(t_n, y_n) \right) \]
is consistent and stable.

Problem 5.
Show that the region of absolute stability for the trapezoidal method is the set of all complex $h\lambda$ with $\text{Real}(\lambda) < 0$.

Problem 6. Bonus Problem for Extra Credit of 20 points (this problem is not required).
(Computational question): Apply the classical fourth order Runge-Kutta method to the test problem:
\[ y' = \lambda y + (1 - \lambda) \cos(t) - (1 + \lambda) \sin(t), \quad y(0) = 1, \]
for various values of $\lambda$ and $h$. For example, try $\lambda = -1, -10, -50$ and $h = 0.5, 0.1, 0.01$. Discuss the results.
Hint: One can consider intervals for $x$, like $[0, 1]; [0, 2]; [0, 3]; [0, 4]; [0, 5]$. 


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