On the Solution of the Elliptic Interface Problems by Difference Potentials Method

Yekaterina Epshteyn and Michael Medvinsky

Abstract Designing numerical methods with high-order accuracy for problems in irregular domains and/or with interfaces is crucial for the accurate solution of many problems with physical and biological applications. The major challenge here is to design an efficient and accurate numerical method that can capture certain properties of analytical solutions in different domains/subdomains while handling arbitrary geometries and complex structures of the domains. Moreover, in general, any standard method (finite-difference, finite-element, etc.) will fail to produce accurate solutions to interface problems due to discontinuities in the model's parameters/solutions. In this work, we consider Difference Potentials Method (DPM) as an efficient and accurate solver for the variable coefficient elliptic interface problems.

1 Introduction

In this paper, we consider Difference Potentials Method (DPM) as an efficient and accurate solver for variable coefficient elliptic interface problems. DPM can be understood as the discrete version of the method of generalized Calderon's potentials and Calderon's boundary equations with projections in the theory of partial differential equations (PDEs). DPM introduces a computationally simple auxiliary domain. The original domain of the problem is embedded into an auxiliary domain, and the auxiliary domain is discretized using simple structured grids, e.g. Cartesian grids. After that, the main idea of DPM is to define a Difference Potentials operator, and to reformulate the original discretized PDEs (without imposed boundary/interface conditions yet) as equivalent discrete generalized Calderon's boundary equations with projections (BEP). These BEP are supplemented by the given boundary/interface

Yekaterina Epshteyn
Department of Mathematics, The University Of Utah, e-mail: epshteyn@math.utah.edu

Michael Medvinsky
Department of Mathematics, The University Of Utah, e-mail: mmedvin@math.utah.edu

conditions (the resulting BEP are always well-posed, as long as the original problem is well-posed), and solved to obtain the values of the solution at the points near the continuous boundary of the original domain (at the points of the discrete grid boundary which approximates the continuous boundary from the inside and outside of the domain). Using the obtained values of the solution at the discrete grid boundary, the approximation to the solution in the original domain is constructed through the discrete generalized Green's formula. *DPM offers geometric flexibility (without the use of unstructured meshes or "body-fitted" meshes), but does not require explicit knowledge of the fundamental solution, is not limited to constant coefficient problems or linear problems, does not involve singular integrals, and can handle general boundary and/or interface conditions.* The reader can consult [18] and [14, 15] for a detailed theoretical study of the methods based on Difference Potentials, and ([18, 16, 21, 12, 11, 13, 8, 20, 19, 17, 4, 7, 6, 1], etc.) for the recent developments and applications of DPM.

In this paper, we extend the work on DPM for the elliptic interface problems started in [19, 17, 6] to variable coefficient elliptic interface models in 2D. A more detailed presentation of DPM for elliptic (and parabolic interface problems) in 2D with different high-order accurate discretizations, as well as the analysis of DPM for the interface problems will be part of the future publications [5], [2].

The paper is organized as follows. In Section 2, we introduce the formulation of the problem. Next, in Section 2.1 we briefly describe the main building blocks of the DPM. Finally, we illustrate the performance of the proposed DPM, as well as compare DPM with the Mayo's method [10], [3] and the Immersed Interface Method (IIM) [9], [3] in several challenging numerical experiments (performed by M. Medvinsky) in Section 2.2.

2 Elliptic Interface Problem

In this work we consider interface/composite domain problem defined in some bounded domain $D^0 \subset \mathbb{R}^2$:

$$L_D u = \begin{cases} L_1 u_{D_1} = f_1(x,y) & (x,y) \in D_1 \\ L_2 u_{D_2} = f_2(x,y) & (x,y) \in D_2 \end{cases} \tag{1}$$

subject to the appropriate interface conditions:

$$u_{\overline{D}_1}\Big|_{\Gamma} - u_{\overline{D}_2}\Big|_{\Gamma} = \phi_1(x,y), \qquad \frac{\partial u_{\overline{D}_1}}{\partial n}\bigg|_{\Gamma} - \frac{\partial u_{\overline{D}_2}}{\partial n}\bigg|_{\Gamma} = \phi_2(x,y) \tag{2}$$

and boundary conditions

$$u|_{\partial D} = \psi(x,y) \tag{3}$$

where $D_1 \cup D_2 = D$ and $D \subset D^0$, see Fig. 1. Here, we assume L_s , $s \in \{1,2\}$ are the second-order linear elliptic differential operators of the form

$$L_s u_{D_s} \equiv \frac{\partial}{\partial x}\left(a_s(x,y)\frac{\partial u_{D_s}}{\partial x}\right) + \frac{\partial}{\partial y}\left(b_s(x,y)\frac{\partial u_{D_s}}{\partial y}\right), \quad s \in \{1,2\}.$$

The functions $a_s(x,y) \geq 1$ and $b_s(x,y) \geq 1$ are sufficiently smooth and defined in a larger auxiliary subdomains $D_s \subset D_s^0$.The functions $f_s(x,y)$ are sufficiently smooth functions defined in each subdomain D_s. We assume that the continuous problem (1)-(3) is well-posed. *Moreover, we assume that the operators L_s are well-defined on some larger auxiliary domain D_s^0. More precisely, we assume that for any sufficiently smooth functions $f_s(x,y)$ the equations $L_s u_{D_s^0} = f_s(x,y)$ have a unique solution $u_{D_s^0}$ on D_s^0 that satisfy the given boundary conditions on ∂D_s^0.*

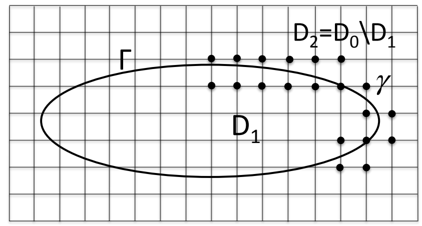


Fig. 1 Example of an auxiliary domain D^0, original domains D_1 and D_2 separated by the interface Γ, and the example of the points in the discrete grid boundary set γ for the 5-point stencil of the second-order method. Auxiliary domain D^0 coincides with D here.

Note, here and below, the upper/or lower index $s \in \{1,2\}$ is introduced to distinguish between the subdomains.

2.1 *Difference Potentials Method for Interface/Composite Domain Problems*

Here we discuss the development of high-order methods based on Difference Potentials approach for the elliptic interface/composite domain problem (1)-(3). Below, we only briefly discuss main ideas of DPM for interface problems. The reader can consult [18, 19, 17, 6, 1] and future publications [5, 2] for more details. Also, the reader can consult [18] for the detailed discussion on the general theory and numerical analysis of DPM. Let us briefly describe the main steps of the algorithm.

Introduction of the Auxiliary Domain: Place the original domains D_s, $s \in \{1,2\}$ in the auxiliary computationally simple domains $D_s^0 \subset \mathbb{R}^2$ that we will choose to be squares. Next, introduce a Cartesian mesh for each D_s^0, with points $x_j^s = j\Delta x^s, y_k^s = k\Delta y^s, (k, j = 0, \pm 1, ...)$. Let us assume for simplicity that $\Delta x^s = \Delta y^s := h^s$. Select discretization of the continuous model (1), for example here we will consider a finite-difference approximation. Next, define a finite-difference stencil $N_{j,k}^s$ with its center placed at (x_j^s, y_k^s) (like a 5 node "dimension by dimension stencil" for the second-order scheme, or a 9 node "dimension by dimension stencil" for the classical fourth-order scheme, etc.). Additionally, introduce the point sets M_s^0 (the set of all the mesh nodes (x_j^s, y_k^s) that belong to the interior of the auxiliary domain D_s^0), $M_s^+ := M_s^0 \cap D_s$ (the set of all the mesh nodes (x_j^s, y_k^s) that belong to the interior of the original domain D_s), and by $M_s^- := M_s^0 \backslash M_s^+$ (the set of all the mesh nodes (x_j^s, y_k^s) that are inside of the auxiliary domain D_s^0 but don't belong to the interior of the original domain D_s). Define $N_s^+ := \{\bigcup_{j,k} N^s_{j,k} | (x^s_j, y^s_k) \in M_s^+\}$ (the set of all points covered by the stencil $N^s_{j,k}$ when center point (x^s_j, y^s_k) of the stencil goes through all the points of the set $M_s^+ \subset D_s$). Similarly, define $N_s^- := \{\bigcup_{j,k} N^s_{j,k} | (x_j, y_k) \in M_s^-\}$ (the set of all points covered by the stencil $N^s_{j,k}$ when center point (x^s_j, y^s_k) of the stencil goes through all the points of the set M_s^-).

Introduce $\gamma_s := N_s^+ \cap N_s^-$. The set γ_s is called the *discrete grid boundary*. The mesh nodes from set γ_s straddle the boundary ∂D_s. $N_s^0 := \{\bigcup_{j,k} N^s_{j,k} | (x^s_j, y^s_k) \in M_s^0\} \subset \overline{D^0_s}$. The sets N_s^0, M_s^0, N_s^+, N_s^-, M_s^+, M_s^-, γ_s will be used to develop the method based on the Difference Potentials approach, Fig. 1.

Difference Equations: The discrete reformulation of the model problem (1) in each auxiliary domain D^0_s is: solve for $u^s_{j,k} \in N_s^+$

$$L^s_h[u^s_{j,k}] = F^s_{j,k}, \quad (x^s_j, y^s_k) \in M_s^+ \tag{4}$$

where $L^s_h[u^s_{j,k}]$ is the discrete linear elliptic operator obtained using finite-difference approximation of order r (for example, the second-order $r = 2$ or the fourth-order $r = 4$, etc.). $F^s_{j,k}$ denotes the discrete right-hand side. The unknowns are $u^s_{j,k} :\approx u_{D_s}(x^s_j, y^s_k)$, where (x^s_j, y^s_k) is a mesh point of the Cartesian grid.

We need to complete the linear system of difference equations (4) with the appropriate choice of the numerical boundary and interface conditions to construct a unique accurate approximation of the continuous problem (1)-(3) in domain D. Thus, to design an efficient algorithm for any type of boundary and interface conditions, we will consider a numerical method based on the idea of the Difference Potentials.

Step 1: Construction of a Particular Solution: Denote by $u^s_{j,k} := G^h_s F^s_{j,k}, \quad u^s_{j,k} \in N_s^+$ the particular solution of the discrete problem (4), which we will construct as the solution (restricted to set N_s^+) of the simple auxiliary problem (AP) of the following form:

$$L^s_h[u^s_{j,k}] = \begin{cases} F^s_{j,k}, & (x^s_j, y^s_k) \in M_s^+, \\ 0, & (x^s_j, y^s_k) \in M_s^-, \end{cases} \tag{5}$$

$$u^s_{j,k} = 0, \quad (x^s_j, y^s_k) \in N^0_s \backslash M^0_s \tag{6}$$

Step 2: Difference Potentials and Construction of the BEP: We now introduce a linear space $\mathbf{V}_{\gamma_s}$ of all the grid functions denoted by v_{γ_s} defined on γ_s [18], [19, 17, 6], etc. We will extend the value v_{γ_s} by zero to other points of the grid N^0_s.

Definition 1. The Difference Potential with any given density $v_{\gamma_s} \in \mathbf{V}_{\gamma_s}$ is the grid function $u^s_{j,k} := \mathbf{P}_{N^+\gamma_s} v_{\gamma_s}$, defined on N_s^+, and coincides on N_s^+ with the solution $u^s_{j,k}$ of the simple auxiliary problem (AP) of the following form:

$$L^s_h[u^s_{j,k}] = \begin{cases} 0, & (x^s_j, y^s_k) \in M_s^+, \\ L^s_h[v_{\gamma_s}], & (x^s_j, y^s_k) \in M_s^-, \end{cases} \tag{7}$$

$$u^s_{j,k} = 0, \quad (x^s_j, y^s_k) \in N^0_s \backslash M^0_s \tag{8}$$

Here, $\mathbf{P}_{N^+\gamma_s}$ denotes the operator which constructs the Difference Potential $u^s_{j,k} = \mathbf{P}_{N^+\gamma_s} v_{\gamma_s}$ from the given density $v_{\gamma_s} \in V_{\gamma_s}$. The operator $\mathbf{P}_{N^+\gamma_s}$ is the linear operator of the density v_{γ_s}. Hence, it can be easily constructed [19, 17, 6]. We will now state the most important theorem of the method:

Theorem 1. *Density u_{γ_s} is the trace of some solution $u^s_{j,k} \in N^+_s$ to the Difference Equations (4) : $u_{\gamma_s} \equiv Tr_{\gamma_s} u^s_{j,k}$, if and only if, u_{γ_s} satisfies Generalized Calderon's Boundary Equations with Projections (BEP)*

$$u_{\gamma_s} - \mathbf{P}_{\gamma_s} u_{\gamma_s} = G^h_s F_{\gamma_s}, \tag{9}$$

where $G^h_s F_{\gamma_s} := Tr_{\gamma_s}(G^h_s F^s_{j,k})$ is the trace (or restriction) of the particular solution $G^h_s F^s_{j,k} \in N^+_s$ constructed in (5)-(6) on the grid boundary γ_s, and $\mathbf{P}_{\gamma_s} u_{\gamma_s} := Tr_{\gamma_s}(\mathbf{P}_{N^+\gamma_s} u_{\gamma_s})$ is the trace of the Difference Potential $\mathbf{P}_{N^+\gamma_s} u_{\gamma_s} \in N^+_s$ in (7)-(8) on the grid boundary γ_s.

Remark: The BEP (9) are constructed for each subdomain and solved efficiently together with the boundary and interface conditions for the unknown densities u_{γ_s} using the idea of the extension operator for u_{γ_s}, and the spectral approach for the approximation of the Cauchy data $(u^s, \frac{\partial u^s}{\partial n})|_{\partial D_s}$ ([19, 17, 12], etc.).

Step 3: Construction of the Approximate Solution to the Model Problem (1)-(3) from the density u_{γ_s} obtained in Step 2:

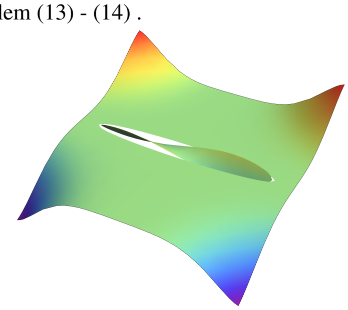
Statement 1 *(Generalized Green's Formula)*
The discrete solution $u^s_{j,k} := \mathbf{P}_{N^+\gamma_s} u_{\gamma_s} + G^h_s F^s_{j,k}$ is the approximation to the solution $u^s_{j,k} \approx u^s(x^s_j, y^s_k)$, $(x^s_j, y^s_k) \in N^+_s \cap D_s$ of the continuous problem (1)-(3) (see [15, 14, 18] for a general theory of DPM and [19, 17, 6, 1, 5]).

The expected accuracy of the proposed method for domains with the smooth boundaries and under sufficient regularity of the exact solutions will be $O(h^{r-\varepsilon})$ in the discrete Hölder norm of order $2+\varepsilon$ (if the continuous second-order linear elliptic operator L is approximated with r^{th} order of accuracy by the discrete operator L_h, and the extension operator for u_{γ_s} is constructed with sufficient accuracy), see [15, 14, 18], [19, 17, 6, 1, 5] and Section 2.2. Here, ε is an arbitrary number with $0 < \varepsilon < 1$.

2.2 Numerical Examples

In the numerical examples below, we consider a second-order centered finite-difference approximation (with 5-node stencil) as the underlying discretization for DPM. The numerical experiments for the fourth-order approximation will be presented in future publication [5]. The first test problem that we

Fig. 2 Exact solution to the test problem (13) - (14) .



present here is the problem from the paper [3]:

$$\Delta u_{D_s} = f_s(x,y), \quad (x,y) \in D_s, \quad s \in \{1,2\} \tag{10}$$

where the interface between two subdomains D_1 and D_2 (see Fig. 1) is given by an ellipse with semi-axes $(a,b) = (0.9, 0.1)$, and the curvature is $\kappa = -90$ at $(\pm a, 0)$ which leads to a quite challenging tests [3]. The exact solution here is

$$u_1 = \sin x \cos y, \quad u_2 = 0, \tag{11}$$

which is discontinuous at the interface. The results for the test problem (10)-(11) are presented in Table 1, which shows the relative error in the maximum norm of the solution and its derivatives. To match the settings of the numerical experiments in paper [3], we consider auxiliary domains (here and below) $D_1^0 = D_2^0 \equiv D = [-1.1, 1.1] \times [-1.1, 1.1]$ for the subdomains D_1 and D_2 respectively, Fig. 1. Note, that in these settings, $h^1 = h^2 = h$ (however, DPM handles as easily different auxiliary problems/non-matching meshes [19, 17, 7, 6, 1]). As observed from the Table 1 here, and from the Table 1 (bottom), on page 111 in paper [3], the accuracy in the solution for the test problem (10)-(11) obtained by DPM is very close to the accuracy obtained by Mayo's Method and by IIM. But, the accuracy in the derivatives of the solution obtained by DPM is superior to the accuracy obtained by Mayo's Method or IIM.

Table 1 Test problem (10) - (11) with $a = 0.9$, $b = 0.1$ from paper [3]. Here N corresponds to half of the number of subintervals (the same number of subintervals in x and y-direction), similarly to the results in Table 1 (bottom), page 111 in [3]. Relative L_∞ error in the solution and in its derivatives.

N	L_∞-error in u	Rate	L_∞-error in u_x	Rate	L_∞-error in u_y	Rate
40	$1.7474e-06$		$1.0559e-06$		$1.0041e-06$	
80	$5.2910e-07$	1.72	$1.7733e-07$	2.57	$1.6081e-07$	2.64
160	$1.2986e-07$	2.03	$2.5886e-08$	2.78	$2.1461e-08$	2.91
320	$3.1742e-08$	2.03	$1.7307e-09$	3.90	$1.3500e-09$	3.99
640	$7.8701e-09$	2.01	$2.0067e-10$	3.11	$1.3030e-10$	3.37

The second test problem is again from [3] and has the same settings as the first test problem (10)-(11), but now the exact solution is defined as:

$$u_1 = x^9 y^8, \quad u_2 = 0. \tag{12}$$

The results for this test problem are presented in Table 2. DPM errors for this test problem (10), (12) are again close to the errors for Mayo's method and IIM, reported in Table 3, page 113 in [3]. As the last and more challenging test problem, we consider the interface problem with variable coefficients as described below:

Table 2 Test problem (10), (12) with $a = 0.9$, $b = 0.1$ from paper [3]. Here N corresponds to half of the number of subintervals (the same number of subintervals in x and y-direction) , similarly to the results in Table 3, page 113 in [3]. Relative L_∞ error in the solution and its derivatives.

N	L_∞-error in u	Rate	L_∞-error in u_x	Rate	L_∞-error in u_y	Rate
40	$1.0000e+00$		$8.3442e-01$		$1.0000e+00$	
80	$2.6622e-01$	1.91	$2.2263e-01$	1.91	$3.3108e-01$	1.59
160	$3.8645e-02$	2.78	$2.2076e-02$	3.33	$5.0801e-02$	2.70
320	$9.0971e-03$	2.09	$2.7015e-03$	3.03	$7.7708e-03$	2.71
640	$2.3838e-03$	1.93	$3.3376e-04$	3.02	$1.0421e-03$	2.90

$$\frac{\partial}{\partial x}\left(a_s(x,y)\frac{\partial u_{D_s}}{\partial x}\right) + \frac{\partial}{\partial y}\left(b_s(x,y)\frac{\partial u_{D_s}}{\partial y}\right) = f_s(x,y), \quad (x,y) \in D_s, \quad s \in \{1,2\} \tag{13}$$

where $a_1 = (3 + 0.5\sin(2x+y))$ $b_1 = (2 + 0.5\cos(4x+3y))$ and $a_2 = b_2 = 10^6$. The interface curve for this problem is again given by the ellipse with semi-axes $(a,b) = (0.9, 0.1)$. The exact solution for this test problem (13) is set to

$$u_1 = \sin(y^2 x)\sin(x^3 y), \quad u_2 = \sin(2x)\sin(3y). \tag{14}$$

The interface problem (13)-(14) is much more challenging than the previous test problems since it has discontinuous solution at the interface, as well as a large jump ratio between diffusion coefficients in subdomains D_1 and D_2, Fig. 2. The results for this test problem are presented in Table 3, which shows the relative error of the solution and its derivatives in the maximum norm. As in the previous numerical examples, DPM preserves overall second-order (and even slightly better in the derivative) accuracy in the solution and its derivatives. The observed numerically in Tables 1-3 slightly higher order of accuracy in the derivatives could be due to the specifics of the considered test problems and the properties of the extension operator for u_{γ_s}.

Table 3 Test problem (13) - (14) with $a = 0.9$, $b = 0.1$. Here N corresponds to half of the number of subintervals (the same number of subintervals in x and y-direction), similarly to previous examples. Relative L_∞ error in the solution and its derivatives.

N	L_∞-error in u	Rate	L_∞-error in u_x	Rate	L_∞-error in u_y	Rate
40	$4.5671e-04$		$1.3639e-04$		$1.3981e-03$	
80	$1.1520e-04$	1.99	$2.2087e-05$	2.63	$3.1356e-04$	2.16
160	$2.8329e-05$	2.02	$2.3138e-06$	3.25	$3.5176e-05$	3.16
320	$7.0319e-06$	2.01	$3.1931e-07$	2.86	$4.6670e-06$	2.91
640	$1.7578e-06$	2.00	$4.9421e-08$	2.69	$7.2111e-07$	2.69

Acknowledgements We are grateful to Jason Albright and Kyle R. Steffen for the comments that helped to improve the manuscript. The research of Yekaterina Epshteyn and Michael Medvinsky is supported in part by the National Science Foundation Grant # DMS-1112984.

References

1. J. Albright, Y. Epshteyn, and K.R. Steffen. High-order accurate difference potentials methods for parabolic problems. *Applied Numerical Mathematics*, 93:87–106, July 2015. http://dx.doi.org/10.1016/j.apnum.2014.08.002.
2. J. Albright, Y. Epshteyn, and Q. Xia. High-order difference potentials methods for 2D parabolic interface problems. April 2015. work in progress.
3. J. Thomas Beale and Anita T. Layton. On the accuracy of finite difference methods for elliptic problems with interfaces. *Commun. Appl. Math. Comput. Sci.*, 1:91–119 (electronic), 2006.
4. Y. Epshteyn. Upwind-difference potentials method for Patlak-Keller-Segel chemotaxis model. *Journal of Scientific Computing*, 53(3):689 – 713, 2012.
5. Y. Epshteyn and M. Medvinsky. On the analysis of DPM for the elliptic interface problems. 2015. in preparation.
6. Y. Epshteyn and S. Phippen. High-order difference potentials methods for 1D elliptic type models. *Applied Numerical Mathematics*, 93:69–86, July 2015. http://dx.doi.org/10.1016/j.apnum.2014.02.005.
7. Yekaterina Epshteyn. Algorithms composition approach based on difference potentials method for parabolic problems. *Commun. Math. Sci.*, 12(4):723–755, 2014.
8. E. Kansa, U. Shumlak, and S. Tsynkov. Discrete Calderon's projections on parallelepipeds and their application to computing exterior magnetic fields for FRC plasmas. *J. Comput. Phys.*, 234:172–198, 2013. http://dx.doi.org/10.1016/j.jcp.2012.09.033.
9. Randall J. LeVeque and Zhi Lin Li. The immersed interface method for elliptic equations with discontinuous coefficients and singular sources. *SIAM J. Numer. Anal.*, 31(4):1019–1044, 1994.
10. Anita Mayo. The fast solution of Poisson's and the biharmonic equations on irregular regions. *SIAM J. Numer. Anal.*, 21(2):285–299, 1984.
11. M. Medvinsky. High order numerical simulation of waves using regular grids and non-conforming interfaces. *Ph.D. Dissertation Tel Aviv University*, 2013.
12. M. Medvinsky, S. Tsynkov, and E. Turkel. The method of difference potentials for the Helmholtz equation using compact high order schemes. *J. Sci. Comput.*, 53(1):150–193, 2012.
13. M. Medvinsky, S. Tsynkov, and E. Turkel. High order numerical simulation of the transmission and scattering of waves using the method of difference potentials. *J. Comput. Phys.*, 243:305–322, 2013.
14. A. A. Reznik. Approximation of surface potentials of elliptic operators by difference potentials. *Dokl. Akad. Nauk SSSR*, 263(6):1318–1321, 1982.
15. A. A. Reznik. Approximation of surface potentials of elliptic operators by difference potentials and solution of boundary value problems. *Ph.D, Moscow, MPTI*, 1983.
16. V. S. Ryaben′kiĭ. Difference potentials analogous to Cauchy integrals. *Uspekhi Mat. Nauk*, 67(3(405)):147–172, 2012.
17. V. S. Ryaben′kiĭ, V. I. Turchaninov, and E. Yu. Èpshteĭn. An algorithm composition scheme for problems in composite domains based on the method of difference potentials. *Zh. Vychisl. Mat. Mat. Fiz.*, (10):1853–1870.
18. Viktor S. Ryaben′kii. *Method of difference potentials and its applications*, volume 30 of *Springer Series in Computational Mathematics*. Springer-Verlag, Berlin, 2002.
19. V.S. Ryaben'kii, V. I. Turchaninov, and Ye. Yu. Epshteyn. The numerical example of algorithms composition for solution of the boundary-value problems on compound domain based on difference potential method. *Moscow, Keldysh Institute for Applied Mathematics, Russia Academy of Sciences*, (3), 2003. http://library.keldysh.ru/preprint.asp?lg=&id=2003-3.
20. V.S. Ryaben'kii and S. Utyuzhnikov. An algorithm of the method of difference potentials for domains with cuts. *Applied Numerical Mathematics*, 93:254–261, 2015.
21. S. V. Utyuzhnikov. Nonlinear problem of active sound control. *J. Comput. Appl. Math.*, 234(1):215–223, 2010.