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Algorithm Composition Scheme for Problems in Composite Domains Based on the Difference Potential Method

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Abstract—An algorithm composition scheme for the numerical solution of boundary value problems in composite domains is proposed and illustrated using an example. The scheme requires neither difference approximations of the boundary conditions nor matching conditions on the boundary between the subdomains. The scheme is suited for multiprocessor computers.

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INTRODUCTION

An algorithm composition scheme for the numerical solution of boundary value problems in composite and complex domains is proposed and illustrated using an example. The scheme is based on the difference potential method and is suitable for multiprocessor computers.

An example of a problem in a composite domain is that of determining the steady-state temperature in a body composed of two materials with different heat conduction properties with given conditions set on the boundary of the body and with matching conditions specified at the interface between the materials constituting the body.

Generally, the various parts of a composite domain can have even different dimensionality (e.g., a three-dimensional body with flat edges). The desired physical fields in the constituent subdomains can be described by different equations (heat conduction in a metal body with cavities filled with a heat-conducting fluid).

It is natural to construct algorithms for homogeneous bodies and, then, to combine them for the original problem in a composite domain.

An example of a problem in a complex domain is heat conduction in a body with narrow bridges near which the temperature varies sharply when the task is to determine the temperature to equally high accuracy throughout the body. In this case, finite difference methods on a regular grid would require very fine meshes in order to provide a good approximation of the solution near the bridges. Another approach is to apply irregular meshes and the finite element method.

An alternative to these approaches in the case of complex domains is to decompose the original domain into subdomains in each of which the solution has roughly identical smoothness properties. After decomposition, the original complex domain can be regarded as a composite one in which an algorithm can be designed as a composition of the algorithms developed for the subdomains. Thus, the decomposition problem can be viewed as a special case of the composition of an algorithm in a composite domain from algorithms designed for solving the problems in the constituent subdomains with matching conditions imposed at the subdomains' interfaces.

Problems of algorithm decomposition and composition in complex and composite domains were addressed by many authors (see, e.g., [1–8] and the references therein).

In most of these studies, after decomposing the computational domain into subdomains, a fairly convenient finite-difference or finite-element approximation is constructed inside each of them (and in a neighborhood of each of them) and grid approximations of matching conditions for solutions are used on their

boundaries. In decomposition problems, for example, overlapping computational subdomains are frequently used and a solution in a composite domain is sought by applying the Schwarz alternating method in order to obtain boundary conditions for the next iteration in a given subdomain.

In this case, most of the computations in each subdomain are performed irrespective of the others, which is well suited for multiprocessor computers and parallel computations.

The algorithm composition scheme proposed in this paper for the numerical solution of problems in composite domains is based on the difference potential method and is related to [9–16].

To be definite, assume that a composite domain D consists of two subdomains D_1 and D_2 and that the problem in $D = D_1 \cup D_2$ is set by two equations

$$Lu_D = \begin{cases} L_1 u_{D_1} = f_{D_1}(x), & x \in D_1, \quad x = (x_1, x_2), \\ L_2 u_{D_2} = f_{D_2}(x), & x \in D_2, \end{cases}$$

with matching conditions for the solutions u_{D_1} and u_{D_2} specified on the boundaries $\Gamma_1 = \partial D_1$ and $\Gamma_2 = \partial D_2$ and with additional boundary conditions.

We construct expressions

$$u_{D_j}(x, c_1^j, \ldots, c_{K_j}^j), \quad j = 1, 2,$$

that depend on a certain number K_j of arbitrary constants $c_1^j, \ldots, c_{K_j}^j$ $(j = 1, 2)$ and somewhat approximate the general solutions u_{D_j} to the original equation in D_j. It is assumed that the approximation becomes progressively more accurate with increasing K_j, $j = 1, 2$.

After an approximation of the general solution

$$u_D = \begin{cases} u_{D_1}(x, c_1^1, \ldots, c_{K_1}^1), & x \in D_1, \\ u_{D_2}(x, c_1^2, \ldots, c_{K_2}^2), & x \in D_2, \end{cases}$$

to the original equation with given right-hand sides was constructed, the desired particular solution is obtained by choosing constants $c_1^1, \ldots, c_{K_1}^1$ and $c_1^2, \ldots, c_{K_2}^2$ for which the boundary and matching conditions are satisfied as accurately as possible.

There are three major difficulties encountered in implementing this plan.

1. For a good approximation, K_1 and K_2 must be sufficiently large.

2. To construct the general solutions u_{D_j}, we have to find K_j particular solutions in the curvilinear domains D_j, $j = 1, 2$.

3. A system of equations of order $K_1 + K_2$ (i.e., a high-order system) has to be solved to determine $c_1^j, \ldots, c_{K_j}^j$, $j = 1, 2$.

The difference potential method makes it possible to reduce these difficulties and design a reasonable, flexible algorithm.

The number $K_1 + K_2$ of arbitrary constants required for achieving a prescribed accuracy in the approximation is decreased using the difference potential method, which involves a nondifference approximation of the boundary and matching conditions that automatically takes into account the smoothness of the solution. Such self-tuning is impossible for difference or finite-element approximations of these conditions. The difference potential method also overcomes the difficulties associated with the computation of solutions to the boundary value problems in the curvilinear domains D_j. Specifically, these problems are reduced to those in simple auxiliary domains with simple auxiliary boundary conditions. The latter are solved using difference schemes on regular grids that do not need to be consistent with each other or with the curvilinear boundaries of D_1 and D_2. Another important point is that the algorithms for computing the solutions to the large number $K_1 + K_2$ of problems for deriving $u_{D_j}(x, c_1^1, \ldots, c_{K_j}^1)$ are entirely independent of each other; therefore, they are perfectly suited for multiprocessor computers.

This paper is organized as follows.

Section 1 presents the necessary preliminaries to potentials with projectors and to difference potentials for their discrete constructive approximations [14].

In Section 2, we outline an abstract general scheme for the composition of an algorithm for second-order elliptic equations based on potentials with projectors and pseudodifferential boundary equations. Later, this abstract scheme is discretized and made constructive by applying the difference potential method.

Section 2 is a development of the scheme sketched in [15, ch. 13].

A numerical example illustrating the general approach is presented in Section 3. The effect of the various parameters of a particular algorithm is estimated qualitatively and quantitatively.

1. PRELIMINARIES TO POTENTIALS WITH PROJECTORS [13–15]

Our considerations are restricted to constructions for second-order linear elliptic equations

$$Lu = f(x_1, x_2), \quad (x_1, x_2) \in D,$$

in a bounded domain D and its neighborhood Q, $D \subset Q$. Examples of such equations are

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} = f(x_1, x_2), \quad (x_1, x_2) \in D,$$

$$\frac{\partial}{\partial x_1}\left[a(x_1, x_2)\frac{\partial u}{\partial x_1}\right] + \frac{\partial}{\partial x_2}\left[b(x_1, x_2)\frac{\partial u}{\partial x_2}\right] = f(x_1, x_2), \quad (x_1, x_2) \in D.$$

Here, $a(x_1, x_2) \geq 1$, $b(x_1, x_2) \geq 1$, and $f(x_1, x_2)$ are given functions that are smooth in the given domain $\overline{Q}$, in which D is contained strictly inside together with its piecewise smooth boundary $\Gamma = \partial D$.

The Cauchy data v_Γ for an arbitrary continuous piecewise smooth function $v(x_1, x_2)$ defined on Γ and in some of its neighborhoods are defined as

$$v_\Gamma = \begin{pmatrix} v|_\Gamma \\ \left.\dfrac{\partial v}{\partial n}\right|_\Gamma \end{pmatrix}, \tag{1}$$

where $\dfrac{\partial}{\partial n}$ is the inward normal derivative relative to D.

Consider the auxiliary problem

$$Lu = f(x_1, x_2), \quad (x_1, x_2) \in Q, \tag{2}$$

with u vanishing on the boundary of $\overline{Q}$:

$$u|_{\partial Q} = 0. \tag{3}$$

Since Q containing D can be arbitrary, for convenience, we choose Q to be a square. It is well known that the Dirichlet boundary value problem (2), (3) for a second-order linear elliptic equation is uniquely solvable.

Let us construct a potential with a density v_Γ. Define the vector function

$$v_\Gamma = \begin{pmatrix} \varphi^0(s) \\ \varphi^1(s) \end{pmatrix}, \tag{4}$$

where $\varphi^0(s)$ and $\varphi^1(s)$ are two piecewise smooth continuous functions on $|\Gamma|$ that are s-periodic with a period of $|\Gamma|$, s is the arc length along Γ, and $|\Gamma|$ is the length of the boundary. Here, the arc length is chosen as a parameter only for definiteness. Another parameter along Γ is used in the numerical examples given in Section 3.

Let $v(x_1, x_2) = v_Q$ be an arbitrary sufficiently smooth function on Q that satisfies condition (3). Suppose that its Cauchy data v_Γ given by (1) coincide with the vector function v_Γ defined by formula (4).

Consider problem (2), (3) with the right-hand side

$$f(x_1, x_2) = \begin{cases} 0, & (x_1, x_2) \in D^+, \\ Lv, & (x_1, x_2) \in D^- = Q \backslash D^+. \end{cases}$$

The restriction of the solution of this problem to D^+ is denoted by $u_{D^+} = P_{D^+\Gamma} v_\Gamma$. Additionally, consider problem (2), (3) with the right-hand side

$$f(x_1, x_2) = \begin{cases} Lv, & (x_1, x_2) \in D^+, \\ 0, & (x_1, x_2) \in D^-. \end{cases}$$

The restriction of the solution of this problem to D^- is denoted by $u_{D^-} = P_{D^-\Gamma} v_\Gamma$.

We introduce the notation $u_D = P_{D\Gamma} v_\Gamma$, which means $u_{D^+} = P_{D^+\Gamma} v_\Gamma$ or $u_{D^-} = P_{D^-\Gamma} v_\Gamma$ depending on which of the two equalities, $D = D^+$ or $D = D^-$, takes place.

Theorem 1. *The function $u_D = P_{D\Gamma} v_\Gamma$ depends only on Cauchy data* (4) *but is independent of the choice of a particular function $v(x)$ satisfying* (3) *whose Cauchy data coincide with* (4).

By virtue of this theorem, the following concept is defined.

Definition 1. The function $u_{D^+} = P_{D^+\Gamma} v_\Gamma$ is called a potential with the density v_Γ.

Theorem 2. *Suppose that v_Γ coincides with the Cauchy data*

$$u_\Gamma = \begin{pmatrix} u|_\Gamma \\ \left.\dfrac{\partial u}{\partial n}\right|_\Gamma \end{pmatrix}$$

for a solution $u(x_1, x_2)$ to the homogeneous equation $Lu = 0$ $(x \in D)$ that is continuous, together with its first derivative, in the closed domain $\overline{D}$. Suppose also that $u(x_1, x_2)$ satisfies condition (3) *if $D = D^-$. Then, the solution u_D can be recovered everywhere in D from its Cauchy data u_Γ by the formula*

$$u_D = P_{D\Gamma} u_\Gamma. \tag{5}$$

Define a boundary operator P_Γ that maps an arbitrary sufficiently smooth density v_Γ of type (4) from the space V_Γ of all such densities to a pair of functions $u_\Gamma \in V_\Gamma$ by the formula

$$u_\Gamma = P_\Gamma v_\Gamma = \begin{pmatrix} P_{D\Gamma} v_\Gamma|_\Gamma \\ \left.\dfrac{\partial}{\partial n}(P_{D\Gamma} v_\Gamma)\right|_\Gamma \end{pmatrix}.$$

Theorem 3. *Let v_Γ be a vector function of type* (4) *and $u_D(x_1, x_2)$ be a smooth* (*up to the boundary*) *solution to the homogeneous equation $Lu = 0$, $(x_1, x_2) \in D$ that satisfies condition* (3) *if $D = D^-$. Then, v_Γ is Cauchy data for $u_D(x_1, x_2)$ if and only if v_Γ satisfies*

$$v_\Gamma = P_\Gamma v_\Gamma. \tag{6}$$

The operator P_Γ is a projector: $P_\Gamma \equiv P_\Gamma^2$.

This theorem means that the equation $Lu = 0$, $(x_1, x_2) \in D$ for smooth (up to the boundary Γ) solutions that satisfy condition (3) if $D = D^-$ is equivalent to Eq. (6) with respect to the Cauchy data for the solution, and two Eqs. (6) for two scalar functions $v|_\Gamma$ and $\left.\dfrac{\partial v}{\partial n}\right|_\Gamma$ are boundary equations. The latter are pseudodifferential boundary equations with the projector P_Γ constructed for the special second-order elliptic equation under consideration.

Note that the solution to the equation $Lu = 0$, $(x_1, x_2) \in D$ that satisfies (3) if $D = D^-$ with any additional condition $lu_D|_\Gamma = \psi$ specified on Γ is equivalent to the solution of the problem

$$Q_\Gamma u_\Gamma \equiv u_\Gamma - P_\Gamma u_\Gamma = 0, \quad l(P_{D\Gamma} u_\Gamma) = \psi$$

on Γ with respect to the Cauchy data u_Γ for the desired solution.

Remark. The potential $u_D = P_{D\Gamma} v_\Gamma$ is a modification [13] of the Calderon potential [17]. In contrast to the Calderon potential, u_D admits a finite-dimensional constructive approximation by difference potentials constructed for this purpose.

Difference Potentials and Difference Boundary Projectors

In the plane $x_1 O x_2$, we introduce a square grid $x_1 = m_1 h$, $x_2 = m_2 h$ ($m_1, m_2 = 0, \pm 1, \ldots$) with a mesh size h such that $1/h$ is an integer.

Suppose that D lies inside a square Q whose sides are parallel to the coordinate axes and lie on grid lines. Each point $(m_1 h, m_2 h) = m$ is assigned a set N_m that is a five-point finite-difference stencil:

$$n = (m_1 h, m_2 h), \quad n = ((m_1 \pm 1)h, m_2 h), \quad n = (m_1 h, (m_2 \pm 1)h).$$

The second-order differential equation $Lu = f$ is associated with the difference equation

$$\sum_{n \in N_m} a_{mn} u_n = f_m, \quad m \in M \equiv Q_h, \tag{7}$$

(which approximates the former to second-order accuracy in h) with the condition

$$u_n = 0, \quad n \in \partial Q, \tag{8}$$

where Q_h is the set of grid points m lying strictly inside Q, and a_{mn} are the known coefficients. Obviously, the solution $u_N \equiv u_{\overline{Q}_h} = \{u_n\}$ to Eq. (7) is defined on the set $\overline{Q}_h \equiv N = \bigcup N_m$, $m \in M$. It is well known that problem (7), (8) has a unique solution $u_N = \{u_n\}$, $n \in N$ for any right-hand side $f_M = \{f_m\}$, $m \in M$.

The grid analogues of D^+ and D^-, their closures $\overline{D}^+$ and $\overline{D}^-$, and the boundary Γ are defined as

$$D_h^+ = \{m \mid m \in D^+\}, \quad D_h^- = \{m \mid m \in D^-\},$$

$$\overline{D}_h^+ = \bigcup N_m, \quad m \in D_h^+, \quad \overline{D}_h^- = \bigcup N_m, \quad m \in D_h^-,$$

$$\Gamma_h = \overline{D}_h^+ \cap \overline{D}_h^-.$$

Let v_{Γ_h} be an arbitrary grid function on Γ_h and v_N be an arbitrary grid function satisfying condition (8) and coinciding with v_{Γ_h} on Γ_h.

Consider problem (7), (8) with the right-hand side

$$f_m^+ = \begin{cases} 0, & m \in D_h^+, \\ \sum a_{mn} v_n, & m \in D_h^-. \end{cases} \tag{9}$$

The restriction of the solution u_N^+ to problem (7)–(9) with N to $\overline{D}_h^+$, where $\overline{D}_h^+ \subset N$, is denoted by

$$u_{\overline{D}_h}^+ = P_{\overline{D}_h \Gamma_h}^+ v_{\Gamma_h}. \tag{10}$$

Consider problem (7), (8) with the right-hand side

$$f_m^- = \begin{cases} \sum a_{mn} v_n, & m \in D_h^+, \\ 0, & m \in D_h^-. \end{cases}$$

The restriction of the solution u_N^- to this problem with N to $\overline{D}_h^-$, where $\overline{D}_h^- \subset N$, is denoted by

$$u_{\overline{D}_h}^- = P_{\overline{D}_h\Gamma_h}^- v_{\Gamma_h}. \tag{11}$$

Define

$$u_{\overline{D}_h} = P_{\overline{D}_h\Gamma_h} v_{\Gamma_h}, \tag{12}$$

which means (10) when $D_h = D_h^+$ and (11) when $D_h = D_h^-$.

Theorem 4. *The functions defined by* (10) *and* (11) *depend only on* v_{Γ_h} *but are independent of the choice of* v_N *satisfying condition* (8) *and coinciding with* v_{Γ_h} *on* Γ_h.

In view of this theorem, the following concept is well defined.

Definition 2. Grid functions (10) and (11) are called difference potentials with the density v_{Γ_h} on the grid domains $\overline{D}_h^+$ and $\overline{D}_h^-$, respectively.

Let V_{Γ_h} denote the linear space of all grid functions v_{Γ_h} that are the restrictions of all the functions v_N satisfying (8) to Γ_h.

The boundary operators $P_{\Gamma_h}^+ : V_{\Gamma_h} \longrightarrow V_{\Gamma_h}$ and $P_{\Gamma_h}^- : V_{\Gamma_h} \longrightarrow V_{\Gamma_h}$ are defined as

$$\begin{aligned} P_{\Gamma_h}^+ v_{\Gamma_h} &= P_{\overline{D}_h^+\Gamma_h}^+ v_{\Gamma_h}\Big|_{\Gamma_h}, \\ P_{\Gamma_h}^- v_{\Gamma_h} &= P_{\overline{D}_h^-\Gamma_h}^- v_{\Gamma_h}\Big|_{\Gamma_h}, \end{aligned} \tag{13}$$

for every $v_{\Gamma_h} \in V_{\Gamma_h}$. Here, the right-hand sides of (13) are the restrictions of difference potentials (10) and (11) to Γ_h. We will also use the notation $P_{\Gamma_h} : V_{\Gamma_h} \longrightarrow V_{\Gamma_h}$ assuming that P_{Γ_h} coincides with $P_{\Gamma_h}^+$ or $P_{\Gamma_h}^-$ depending on the choice of D_h^+ or D_h^- to be the grid domain D_h.

Theorem 5. *The grid function* v_{Γ_h} *on* Γ_h *is the trace* $v_{\Gamma_h} = u_{\Gamma_h}$ *of a solution* $u_{\overline{D}_h}$ *to the homogeneous equation* $\sum a_{mn} u_n = 0$ $(m \in D_h)$ *that also satisfies condition* (8) *in the case* $D_h = D_h^-$ *if and only if* v_{Γ_h} *satisfies*

$$Q_h v_{\Gamma_h} \equiv v_{\Gamma_h} - P_{\Gamma_h} v_{\Gamma_h} = 0.$$

In this case, $u_{\overline{D}_h}$ *can be recovered from its boundary values* $u_{\Gamma_h} = v_{\Gamma_h}$ *by formula* (12).

This theorem allows us to reduce the problem of finding a solution to the equation

$$\sum_{n \in N_m} a_{mn} u_n = 0, \quad m \in D_h,$$

with some additional conditions $l u_{D_h} = \psi$ to the equivalent problem of finding the density u_{Γ_h} of the difference potential $u_{\overline{D}_h} = P_{\overline{D}_h\Gamma_h} u_{\Gamma_h}$, which is true by virtue of the following system of equations for u_{Γ_h}:

$$\begin{aligned} Q_{\Gamma_h} u_{\Gamma_h} &\equiv u_{\Gamma_h} - P_{\Gamma_h} u_{\Gamma_h} = 0, \\ l(P_{\overline{D}_h\Gamma_h} u_{\Gamma_h}) &= \psi. \end{aligned}$$

2. ALGORITHM COMPOSITION SCHEME

2.1. Statement of the Problem in a Composite Domain

Suppose that $D = D_1 \cup D_2$ in the plane (x_1, x_2) consists of two bounded disjoint domains D_1 and D_2 with piecewise smooth boundaries $\Gamma_j = \partial D_j$, $j = 1, 2$.

The task is to find the function

$$u_D(x_1, x_2) = \begin{cases} u_{D_1}(x_1, x_2), & (x_1, x_2) \in D_1, \\ u_{D_2}(x_1, x_2), & (x_1, x_2) \in D_2, \end{cases} \tag{14}$$

where u_{D_j} satisfies the second-order linear elliptic equations

$$L_j u_{D_j} = f_{D_j}, \quad j = 1, 2, \tag{15}$$

and matching and additional conditions on Γ_1 and Γ_2 of the form

$$l(u_{\Gamma_1}, u_{\Gamma_2}) = \varphi. \tag{16}$$

Here, $l(u_{\Gamma_1}, u_{\Gamma_2})$ is an expression that assigns to each pair u_{Γ_1}, u_{Γ_2}, where

$$u_{\Gamma_j} = \begin{pmatrix} u_{D_j}\big|_{\Gamma_j} \\ \dfrac{\partial u_{D_j}}{\partial n}\Big|_{\Gamma_j} \end{pmatrix}, \quad j = 1, 2,$$

an admissible element φ.

Assume that the coefficients of the differential expressions on the left-hand side of (15) are defined everywhere in the plane.

It is assumed that $f_{D_1}(x_1, x_2)$ and $f_{D_2}(x_1, x_2)$ are smooth functions in the entire plane (x_1, x_2) rather than only on D_1 and D_2.

Define the linear spaces U_{D_j}, F_{D_j}, V_{Γ_j}, and Φ of sufficiently smooth functions u_{D_j}, f_{D_j}, Cauchy data v_{Γ_j}, and right-hand sides φ from conditions (16), respectively. Define the norms

$$\|u_{D_j}\|^2_{U_{D_j}} = \iint\limits_{D_j} \left[u^2_{D_j} + \left(\frac{\partial u_{D_j}}{\partial x_1}\right)^2 + \left(\frac{\partial u_{D_j}}{\partial x_2}\right)^2 \right] dx_1 dx_2, \tag{17a}$$

$$\|f_{D_j}\|^2_{F_{D_j}} = \iint\limits_{D_j} f^2_{D_j} dx_1 dx_2, \tag{17b}$$

$$\|u_{\Gamma_j}\|^2_{V_{\Gamma_j}} = \int\limits_{\Gamma_j} \left[|u^0|^2 + \alpha\left(\left|\frac{du^0}{ds}\right|^2 + |u^1|^2 \right) \right] ds \tag{17c}$$

and some norm

$$\|\varphi\|_\Phi, \quad \varphi \in \Phi.$$

In (17c), α denotes a nonnegative numerical parameter.

Assume that the operator l in (16) and the space Φ with a Hilbert norm are specified so that problem (15), (16) has the same solution u_D for any $f_{D_j} \in F_{D_j}$ $(j = 1, 2)$ and $\varphi \in \Phi$, and it holds that

$$\|u_D\|^2 = \|u_{D_1}\|^2_{U_{D_1}} + \|u_{D_2}\|^2_{U_{D_2}} \le c_1 \|f_{D_1}\|^2_{F_{D_1}} + c_2 \|f_{D_2}\|^2_{F_{D_2}} + c_3 \|\varphi\|^2_\Phi.$$

Additionally, assume that

$$\|l(u_{\Gamma_1}, u_{\Gamma_2})\|_\Phi = \|\varphi\|_\Phi \le c_4 (\|u_{\Gamma_1}\|_{V_{\Gamma_1}} + \|u_{\Gamma_2}\|_{V_{\Gamma_2}}),$$

where c_1, c_2, c_3, and c_4 are constants independent of f_{D_1}, f_{D_2}, φ, u_{Γ_1}, or u_{Γ_2}.

2.2. Preliminary Composition Scheme

The preliminary scheme is based on the general solution to Eq. (15), i.e., on

$$u_{D_j} = P_{D_j\Gamma_j} v_{\Gamma_j} + \bar{u}_{D_j}, \quad j = 1, 2, \tag{18}$$

which was written using the potential

$$u_{\bar{D}_j} = P_{\bar{D}_j\Gamma_j} v_{\Gamma_j}$$

with a density v_{Γ_j} subject to the only condition

$$Q_{\Gamma_j} v_{\Gamma_j} \equiv v_{\Gamma_j} - P_{\Gamma_j} v_{\Gamma_j} = 0, \quad j = 1, 2. \tag{19}$$

The function $\bar{u}_{D_j}$ is any fixed particular solution to the inhomogeneous differential equation (15). Under condition (19), any particular solution to Eq. (15) is derived from general solution (18) with a suitable choice of v_{Γ_j}.

Note that (18) and (19) imply

$$u_{\Gamma_j} = v_{\Gamma_j} + \bar{u}_{\Gamma_j}, \tag{20}$$

where $\bar{u}_{\Gamma_j}$ is the Cauchy data of the particular solution $\bar{u}_{D_j}$.

The desired solution to the original problem (14)–(16) is obtained in form (18) or (20) if v_{Γ_j} is determined by conditions (16) and (19). Here, (16) takes the form

$$l(v_{\Gamma_1} + \bar{u}_{\Gamma_1}, v_{\Gamma_2} + \bar{u}_{\Gamma_2}) = \varphi, \tag{21}$$

and conditions (19) and (21) are jointly considered.

It is well known that Eq. (19) can be solved for the second component of the Cauchy data

$$v^1\big|_{\Gamma_j} = A_j v^0\big|_{\Gamma_j}, \quad j = 1, 2. \tag{22}$$

Here, A_j is called the Poincaré–Steklov operator of the equation

$$L_j v_{D_j} = 0, \quad j = 1, 2.$$

Moreover, $v^0_{\Gamma_j} = v_{D_j}\big|_{\Gamma_j}$ and $v^1_{\Gamma_1} = \left.\dfrac{\partial v_{D_j}}{\partial n}\right|_{\Gamma_j}$.

By using the Poincaré–Steklov operators A_j, $j = 1, 2$, we can replace system (19), (21) with the single equation (21), in which expressions (22) are substituted for the second components of the desired Cauchy data.

2.3. Discretization of Cauchy Data

We fix $j = 1, 2$ and, for each positive integer K_j, specify a set of basis functions

$$\psi_{1j}(s, K_j), \ldots, \psi_{K_j j}(s, K_j)$$

on Γ_j. For simplicity, assume that the basis functions are independent of K_j:

$$\psi_{kj}(s, K_j) = \psi_{kj}(s), \quad k = 1, 2, \ldots, K_j;$$

i.e., the set of basis functions is supplemented with increasing K_j.

For every sufficiently smooth single-valued periodic function $f(s)$ on Γ_j, assume that the sequence

$$\epsilon_{K_j} = \min_{c_k, c'_k} \int [\left|f(s) - \sum c_k \psi_{kj}(s)\right|^2 + \left|f'(s) - \sum c'_k \psi_{kj}(s)\right|^2] ds$$

tends to zero with increasing K_j: $\lim \epsilon_{K_j} = 0$ as $K_j \longrightarrow \infty$.

To discretize the elements

$$v_{\Gamma_j} = \begin{pmatrix} v^0(s) \\ v^1(s) \end{pmatrix} \tag{23}$$

from the space of Cauchy data, we use the approximate equalities

$$\tilde{v}_{\Gamma_j} = \sum_{k=1}^{K_j} [c_k \Psi_{k\Gamma_j} + c_k' \Psi'_{k\Gamma_j}],$$

where

$$\Psi_{k\Gamma_j} = \begin{pmatrix} \psi_{kj}(s) \\ 0 \end{pmatrix}, \quad \Psi'_{k\Gamma_j} = \begin{pmatrix} 0 \\ \psi_{kj}(s) \end{pmatrix}, \tag{24}$$

and c_k and c_k' $(k = 1, 2, \ldots, K_j)$ are the unknown numerical coefficients to be determined.

2.4. Discretization of the Potential

To discretize the potential

$$z_{\overline{D}_j} = P_{\overline{D}_j\Gamma_j} v_{\Gamma_j} \tag{25}$$

with the density given by (23), we use the difference potential

$$z_{\overline{D}_{jh}} = P_{\overline{D}_{jh}\Gamma_{jh}} v_{\Gamma_{jh}},$$

which was constructed above for the difference analogue of the differential equation $L_j v_{D_j} = 0$. As $v_{\Gamma_{jh}}$, we use the function determined by the Cauchy data $\nu \in \Gamma_{jh}$ at each point v_{Γ_j} according to the following two- ($k = 0$) or three-term ($k = 1$) Taylor formula:

$$v_\nu^{(k)} = v^0(s_\nu) + \rho_\nu v^1(s_\nu) + k\frac{\rho_\nu^2}{2} v^{(2)}(s_\nu), \quad \nu \in \Gamma_{jh}. \tag{26}$$

Here, s_ν is the value of the arc length s at the point where Γ_j intersects the normal to Γ_j passing through the point ν. The number ρ_ν is the distance from ν to the intersection point of the normal with Γ_j taken with a plus sign if $\nu \in \overline{D}_j$ and with a minus sign if $\nu \notin D_j$. The function $v^{(2)}(s)$ in (26) is uniquely defined by the formula

$$v^{(2)}(s) = \left.\frac{\partial^2 w}{\partial n^2}\right|_{\Gamma_j},$$

where $w(x_1, x_2)$ is an arbitrary smooth function on Γ_j and in its neighborhood that satisfies

$$w|_{\Gamma_j} = v^0, \quad \left.\frac{\partial w}{\partial n}\right|_{\Gamma_j} = v^1, \quad L_j w|_{\Gamma_j} = 0.$$

Denote by $\pi^{(k)}_{\Gamma_{jh}\Gamma_j}$ the operator that, to every v_{Γ_j} from the space of Cauchy data, assigns $v^{(k)}_{\Gamma_{jh}}$ according to Taylor formula (26), so that

$$v^{(k)}_{\Gamma_{jh}} = \pi^{(k)}_{\Gamma_{jh}\Gamma_j} v_{\Gamma_j}.$$

To approximate (25), we use the difference potential

$$z^{(k)}_{\overline{D}_{jh}} = P_{\overline{D}_{jh}\Gamma_{jh}} [\pi^{(k)}_{\Gamma_{jh}\Gamma_j} v_{\Gamma_j}], \quad k = 0, 1.$$

In the numerical experiment described in Section 3, we compare the Taylor formulas with $k = 0$ and $k = 1$.

2.5. Computation of a Particular Solution and Its Cauchy Data

A particular solution $\bar{u}_{D_j}$ to the inhomogeneous equation $L_j u_{D_j} = f_{D_j}$ included in the general solution $u_{D_j} = P_{D_j\Gamma_j} v_{\Gamma_j} + \bar{u}_{D_j}$ can be obtained as accurately as desired by numerically solving the Dirichlet problem with zero conditions on the boundary in an arbitrary square domain containing D_j. This can be done using a difference scheme on a square grid with a small mesh size. By virtue of the Thome estimates (see, e.g., [14]), the solution to the difference equation and its divided differences converge uniformly to the solution of the differential problem as $h \longrightarrow 0$. Obviously, this allows us to calculate the particular solution $\bar{u}_{D_j}$ and its Cauchy data

$$\bar{u}_{\Gamma_j} = \begin{pmatrix} \bar{u}_{D_j}\big|_{\Gamma_j} \\ \dfrac{\partial \bar{u}_{D_j}}{\partial n}\Big|_{\Gamma_j} \end{pmatrix}$$

as accurately as desired.

2.6. Discrete Norms

The space $V_{\Gamma_{jh}}$ of grid functions $v_{\Gamma_{jh}}$ is equipped by a norm similar to norm (17c) for Cauchy data v_{Γ_j}:

$$\|v_{\Gamma_{jh}}\|^2_{V_{\Gamma_{jh}}} = h\left[\sum |v_\nu|^2 + \alpha\sum \left|\frac{v_{\nu_1+1,\,\nu_2} - v_\nu}{h}\right|^2 + \alpha\sum \left|\frac{v_{\nu_1,\,\nu_2+1} - v_\nu}{h}\right|^2\right]. \tag{27}$$

Here, the sum is extended over all $\nu = (\nu_1, \nu_2)$ that belong to Γ_{jh} in the first term; over all $\nu = (\nu_1, \nu_2)$ that, together with $(\nu_1 + 1, \nu_2)$, belong to Γ_{jh} in the second term; and over all $\nu = (\nu_1, \nu_2)$ that, together with $(\nu_1, \nu_2 + 1)$, belong to Γ_{jh} in the third term over.

2.7. Composition of an Effective Algorithm

Recall that the first step in the algorithm composition scheme for the numerical solution to a problem in a composite domain is to find a general solution to the equation $L_j u_{D_j} = f_j$ of the form

$$u_{D_j} = v_{D_j} + \bar{u}_{D_j}, \quad j = 1, 2, \tag{28}$$

or

$$u_{D_j} = P_{D_j\Gamma_j} v_{\Gamma_j} + \bar{u}_{D_j}, \quad j = 1, 2, \tag{29}$$

where v_{Γ_j} ranges over the solution set of the boundary pseudodifferential equation

$$Q_{\Gamma_j} v_{\Gamma_j} \equiv v_{\Gamma_j} - P_{\Gamma_j} v_{\Gamma_j} = 0, \quad j = 1, 2. \tag{30}$$

Then, we determine a unique pair of Cauchy data

$$v_{\Gamma_1} \in V_{\Gamma_1}, \quad v_{\Gamma_2} \in V_{\Gamma_2} \tag{31}$$

that satisfies Eqs. (30) and the conditions

$$l(v_{\Gamma_1} + \bar{u}_{\Gamma_1}, v_{\Gamma_2} + \bar{u}_{\Gamma_2}) = \varphi. \tag{32}$$

After v_{Γ_j} was found, the desired solution

$$u_D = \begin{cases} u_{D_1}, & (x_1, x_2) \in D_1, \\ u_{D_2}, & (x_1, x_2) \in D_2, \end{cases}$$

to the problem in the composite domain $D = D_1 \cup D_2$ is given by formulas (29).

This scheme is not constructive, since it indicates neither how to solve system (30)–(32) for v_{Γ_1} and v_{Γ_2} nor how to compute the term $v_{D_j} = P_{D_j\Gamma_j} v_{\Gamma_j}$ on the right-hand side of (29). (Recall that the particular solution $\bar{u}_{D_j}$ to the inhomogeneous equation $L_j \bar{u}_{D_j} = f_{D_j}$, as well as its Cauchy data $\bar{u}_{\Gamma_1}$ and $\bar{u}_{\Gamma_2}$, can be regarded as known, since a method for their computation was described in Section 2.5.)

Now, we describe a constructive algorithm. The desired densities v_{Γ_j} ($j = 1, 2$) are represented approximately as

$$v_{\Gamma_j} \approx \tilde{v}_{\Gamma_j} = \sum_{k=1}^{K_j} [c_{kj} \Psi_{k\Gamma_j} + c'_{kj} \Psi'_{k\Gamma_j}], \quad j = 1, 2. \tag{33}$$

For each $j = 1, 2$, Eq. (30) is replaced by the following equation for the $2K_j$ coefficients c_{kj} and c'_{kj}:

$$Q_{\Gamma_{jh}} \tilde{v}_{\Gamma_{jh}} \equiv \tilde{v}_{\Gamma_{jh}} - P_{\Gamma_{jh}} \tilde{v}_{\Gamma_{jh}} = 0, \quad j = 1, 2, \tag{34}$$

where

$$\tilde{v}_{\Gamma_{jh}} = \pi_{\Gamma_{jh}\Gamma_j} \tilde{v}_{\Gamma_j} = \sum_{k=1}^{K_j} [c_{kj} (\pi_{\Gamma_{jh}\Gamma_j} \Psi_{k\Gamma_j}) + c'_{kj} (\pi_{\Gamma_{jh}\Gamma_j} \Psi'_{k\Gamma_j})]. \tag{35}$$

System (34) is a set of linear scalar equations whose order is equal to the number $|\Gamma_{jh}|$ of points on the grid boundary Γ_{jh}. The mesh size h_j is assumed to be so small that $|\Gamma_{jh}| \gg 2K_j$.

Obviously, system (34), (35) can be written in the matrix form

$$B_j \mathbf{c}_j + B'_j \mathbf{c}'_j = 0. \tag{36}$$

Here, $\mathbf{c}_j = (c_{1j}, \ldots, c_{K_j j})$, $\mathbf{c}'_j = (c'_{1j}, \ldots, c'_{K_j j})$, and

$$B_j = (\mathbf{b}_{j1}, \ldots, \mathbf{b}_{jK_j}), \quad B'_j = (\mathbf{b}'_{j1}, \ldots, \mathbf{b}'_{jK_j}),$$

where

$$\mathbf{b}_{jk} = Q_{\Gamma_{jh}} (\pi_{\Gamma_{jh}\Gamma_j} \Psi_{k\Gamma_j}), \quad \mathbf{b}'_{jk} = Q_{\Gamma_{jh}} (\pi_{\Gamma_{jh}\Gamma_j} \Psi'_{k\Gamma_j}), \quad k = 1, 2, \ldots, K_j,$$

are the columns of B_j and B'_j, so that the components of the vectors $\mathbf{b}_{jk}$ and $\mathbf{b}'_{jk}$ are $|\Gamma_j|$ values of the grid functions

$$Q_{\Gamma_{jh}} (\pi_{\Gamma_{jh}\Gamma_j}, \Psi_{k\Gamma_j}), \quad Q_{\Gamma_{jh}} (\pi_{\Gamma_{jh}\Gamma_j}, \Psi'_{k\Gamma_j})$$

taken at points of Γ_{jh} indexed in an arbitrary order.

The weak solution $\tilde{v}_{\Gamma_1}$, $\tilde{v}_{\Gamma_2}$ to system (30)–(32) is defined as Cauchy data of form (33) such that the constants c_{kj} and c'_{kj} minimize the expression

$$\left\| l(\tilde{v}_{\Gamma_1} + \tilde{u}_{\Gamma_1}, \tilde{v}_{\Gamma_2} + \tilde{u}_{\Gamma_2}) - \varphi \right\|_\Phi^2 + \sum_{j=1}^{2} \left\| Q_{\Gamma_{jh}} \pi_{\Gamma_{jh}\Gamma_j} \tilde{v}_{\Gamma_j} \right\|_{V_{\Gamma_{jh}}}^2.$$

For $V_{\Gamma_{jh}}$ ($j = 1, 2$) equipped with Euclidean norms and for Φ equipped with a Hilbert norm, this variational problem is the well-known least squares problem.

After $\tilde{v}_{\Gamma_1}$ and $\tilde{v}_{\Gamma_2}$ were found, the functions $\tilde{v}_{D_1}$ and $\tilde{v}_{D_2}$ involved in (28) are replaced with approximate difference potentials:

$$\tilde{v}_{\bar{D}_{jh}} = P_{\bar{D}_{jh}\Gamma_{jh}} \tilde{v}_{\Gamma_{jh}}, \quad j = 1, 2.$$

By definition, these potentials are calculated by solving difference auxiliary problems of form (7)–(9).

For fixed K_j, $j = 1, 2$, the determined numbers $c_{kj} = c_{kj}(K_j, h_j)$ and $c'_{kj} = c'_{kj}(K_j, h_j)$ converge, as $h_j \longrightarrow 0$, to the limiting values

$$\bar{c}_{kj}(K_j) = \lim_{h_j \to 0} c_{kj}(K_j, h_j), \quad \bar{c}'_{kj}(K_j) = \lim_{h_j \to 0} c'_{kj}(K_j, h_j).$$

It can be shown (see [14, part 1]) that expressions (33) with c_{kj} and c'_{kj} set equal to $\bar{c}_{kj}(K_j)$ and $\bar{c}'_{kj}(K_j)$ tend to the exact solutions of system (30)–(32) as $K_j \longrightarrow \infty$. Moreover, with the natural choice of the basis functions ψ_{kj}, sufficient accuracy is usually achieved even for small K_1 and K_2. Note that an algorithm for computing a solution in $D = D_1 \cup D_2$ based on the general solutions in the constituent domains can be composed by numerous methods, which produce somewhat different approximate solutions that, nevertheless, converge to the exact one as $h_j \longrightarrow 0$ and $K_j \longrightarrow \infty$.

For example, we can proceed from overdetermined system (34) to

$$\mathbf{c}'_j = A_j^{(h)} \mathbf{c}_j,$$

where $A_j^{(h)}$ is defined as a matrix that, to each set of $\mathbf{c}_j$, assigns a set of $\mathbf{c}'_j$ that minimize

$$G_j(\mathbf{c}_j, \mathbf{c}'_j) \equiv \left\| \tilde{v}_{\Gamma_{jh}} - P_{\Gamma_{jh}} \tilde{v}_{\Gamma_{jh}} \right\|^2_{V_{\Gamma_{jh}}} = \left\| B_j \mathbf{c}_j + B'_j \mathbf{c}'_j \right\|^2_{V_{\Gamma_{jh}}}. \tag{37}$$

As is known, this matrix $A_j^{(h)}$ can be calculated by the least squares method. Now, the general solution in each $\overline{D}_{jh}$ is approximated by a difference potential whose density depends only on the set of numbers $\mathbf{c}_j = \{c_{1j}, \ldots, c_{K_j j}\}$.

The operator $A_j^{(h)}$ plays the role of a Poincaré–Steklov operator. To obtain an approximate solution to the original problem in the composite domain, it remains to replace $\mathbf{c}'_j$ in (32) by $A_j^{(h)} \mathbf{c}_j$ and solve the resulting overdetermined system for $\mathbf{c}_j$ ($j = 1, 2$) by the least squares method.

Note that the computation of the matrix of $G_j(\mathbf{c}_j, \mathbf{c}'_j)$ is reduced to the computation of difference potentials with the densities $\pi_{\Gamma_{jh}\Gamma_j} \Psi_{k\Gamma_j}$ and $\pi_{\Gamma_{jh}\Gamma_j} \Psi'_{k\Gamma_j}$, $k = 1, 2, \ldots, K_j$. For this purpose, we need to solve $2K_j$ difference auxiliary problems with different right-hand sides, which is a task well suited for multiprocessor computers. Moreover, the number $2K_j$ of basis functions that is sufficient for achieving the required accuracy of the approximation is usually not very large.

The numbers $\mathbf{c}_j$ and $\mathbf{c}'_j$ ($j = 1, 2$) can be found by iterative algorithms involving repeated solutions of difference auxiliary problems, whose number, however, can turn out to be less than $2(K_1 + K_2)$, i.e., less than the number required for computing B_j and B'_j in (36).

The scheme proposed can be generalized to the case where the equations $L_j u_{D_j} = f_j$ are considered in spatial subdomains and are sets of equations not necessarily of the second order or of the elliptic type.

The possibility of these generalizations is based on that fact that the difference potentials are constructed for general linear systems of difference equations on irregular grids (see [14]).

3. STATEMENT OF TEST PROBLEMS

Let D_1 and D_2 be two disjoint domains whose closures intersect along the curve $\Gamma = \overline{D}_1 \cap \overline{D}_2$ and the union of whose closures is the square $D^0 = \{(x_1, x_2) \in \mathbb{R}^2 \mid -2 \le x_1, x_2 \le 2\}$ (see figure).

On the composite domain $D^0 = \overline{D}_1 \cup \overline{D}_2$, we consider the inhomogeneous boundary value problem

$$L_{D_j} u_{D_j}(x_1, x_2) = f_{D_j}(x_1, x_2), \quad (x_1, x_2) \in D_j, \quad j = 1, 2, \tag{38}$$

with the boundary condition

$$u_{D_1}\big|_{\partial D^0} = u\big|_{\partial D^0} \tag{39}$$

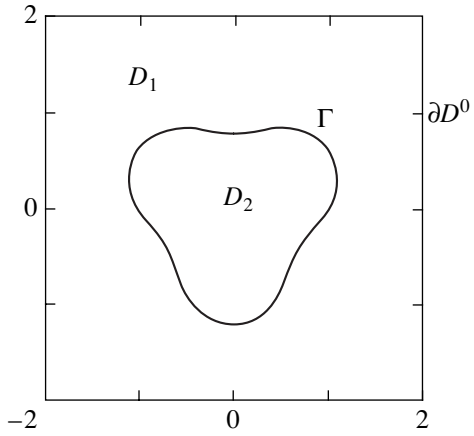

Figure.

and the additional conditions

$$l(u_{\Gamma_1}, u_{\Gamma_2}) \equiv \begin{pmatrix} u_{\bar{D}_1}\big|_\Gamma - u_{\bar{D}_2}\big|_\Gamma \\ \dfrac{\partial u_{\bar{D}_1}}{\partial n}\bigg|_\Gamma - \beta \dfrac{\partial u_{\bar{D}_2}}{\partial n}\bigg|_\Gamma \end{pmatrix} = \varphi, \tag{40}$$

on Γ, where $L_{D_1} = L_{D_2} \equiv \Delta$ is the Laplacian, β is a function on Γ, $\Gamma \equiv \Gamma_1 \equiv \Gamma_2$, $u\big|_{\partial D^0}$ is a given function on ∂D^0, and f_{D_j} and φ are given functions. Here, $\dfrac{\partial}{\partial n}$ denotes the inward (with respect to D_2) normal derivative on Γ.

To construct the general solution u_{D_1}, we use the specific features of problem (38)–(40). The auxiliary square Q_1 is D^0, and the auxiliary problem is

$$\Delta u = f(x_1, x_2), \quad (x_1, x_2) \in D^0 = Q_1,$$

$$u\big|_{\partial Q_1} = 0.$$

As a particular solution $\bar{u}_{D_1}$, we use the restriction to D_1 of the solution u_{Q_1} to the problem

$$\Delta \tilde{u} = f(x_1, x_2), \quad (x_1, x_2) \in D^0 = Q_1,$$

with inhomogeneous boundary condition (39). Here, $f(x_1, x_2)$ is a smooth extension of $f_{D_1}(x_1, x_2)$ to the entire square D^0.

According to the general scheme described in Sections 1 and 2, the computational scheme for the boundary value problem in a composite domain consists of four steps.

Step 1. The subspace spanned by a given finite number of basis functions (24) is distinguished in the space of Cauchy data v_{Γ_j}, $j = 1, 2$. This subspace is associated with quadratic form (37):

$$G_j(\tilde{v}_{\Gamma_j}) = \left\| Q_{\Gamma_{jh}} \pi^{(k)}_{\Gamma_{jh}\Gamma_j} \tilde{v}_{\Gamma_j} \right\|^2_{V_{\Gamma_{jh}}}, \quad j = 1, 2.$$

Step 2. Find any particular solutions $\bar{u}_{D_1}$ to Eqs. (38), (39) and $\bar{u}_{D_2}$ to Eq. (38) and calculate their Cauchy data on Γ_1 and Γ_2, respectively.

Step 3. Solve the joint variational problem; i.e., find two pairs of Cauchy data $\tilde{v}_{\Gamma_j}$ ($j = 1, 2$) that minimize the expression

$$\left\| l(\tilde{v}_{\Gamma_1} + \bar{u}_{\Gamma_1}, \tilde{v}_{\Gamma_2} + \bar{u}_{\Gamma_2}) - \varphi \right\|_{\Phi}^2 + \sum_{j=1,2} G_j(\tilde{v}_{\Gamma_j}).$$

Step 4. Extend the Cauchy data determined to an approximate solution to the equation on a grid inside the domain by using the difference potential $\tilde{v}_{\bar{D}_h} = P_{\bar{D}_h\Gamma_h}\tilde{v}_{\Gamma_h}$, whose density $\tilde{v}_{\Gamma_h}$ is constructed by Taylor formula (26) according to the Cauchy data found.

Note that Steps 1 and 2 are entirely separated from each other. In particular, $G_j(\tilde{v}_{\Gamma_j})$ and $\bar{u}_{\Gamma_j}$ can be calculated using different auxiliary problems with different mesh sizes.

On Γ_j ($j = 1, 2$), we choose the same system of basis functions:

$$\psi_{1j}(s) = 1, \quad \psi_{2j}(s) = \cos\left(\frac{2\pi}{|\Gamma|}s\right), \quad \psi_{3j}(s) = \sin\left(\frac{2\pi}{|\Gamma|}s\right), \quad \ldots, \quad \psi_{2Nj}(s) = \cos\left(\frac{2\pi}{|\Gamma|}Ns\right),$$

$$\psi_{2N+1,\,j}(s) = \sin\left(\frac{2\pi}{|\Gamma|}Ns\right).$$

In the numerical experiments, test problems were solved following the scheme described above. As tests, we used problems with known analytical expressions for the exact solutions. This allowed us to analyze the errors in approximate solutions depending on various parameters (mesh sizes, the number $4N + 2$ of basis functions in V_{Γ_j}, etc.).

Define three auxiliary functions

$$u^{(1)} = \cos(c_1x_1)\cos(c_1x_2),$$

$$u^{(2)} = \sin(c_2x_1)\sin(c_2x_2)P_5(\rho/0.9), \quad \rho = r/r_\Gamma(\theta),$$

$$u^{(3)} = \max\left(3\frac{1-\rho^2}{1+\rho^2}, 0\right),$$

where c_1 and c_2 are regarded as numerical parameters. Here, the Cartesian coordinates (x_1, x_2) of points are related to their polar coordinates (r, θ) by the equality $(x_1, x_2) = r(\cos\theta, \sin\theta)$.

The boundary Γ is parametrically defined in polar coordinates (r, θ) by the relation $r(\theta) = r_\Gamma(\theta) = 1 + e\sin(3\theta)$, where e is a parameter.

The function $P_5(x)$ is continuous; identically equal to 1 for $x \le 0$; vanishes identically for $x \ge 1$; and, on the interval $0 \le x \le 1$, it is a unique ninth-degree polynomial whose derivatives up to the fourth order vanish at the endpoints of $0 \le x \le 1$. The function $P_5(x)$ can be written as

$$P_5(x) = \begin{cases} 1, & x \le 0, \\ 1 - 126x^5 + 420x^6 - 540x^7 + 315x^8 - 70x^9, & 0 \le x \le 1, \\ 0, & x \ge 1. \end{cases}$$

Note that, due to the multiplier $P_5(\rho/0.9)$, the function $u^{(2)}$ vanishes outside D_2 and in a neighborhood of Γ. At the same time, for large values of c_2, $u^{(2)}$ exhibits strong oscillations deep inside D_2. This allowed us to form test problems with widely different smoothness properties in different parts of D_2.

As test problems, we used problems (38)–(40) whose solutions are given by

$$u(x_1, x_2) = \begin{cases} u_{D_1}(x_1, x_2) = u^{(1)}, & (x_1, x_2) \in D_1, \\ u_{D_2}(x_1, x_2) = u^{(1)} + u^{(2)} + c_3u^{(3)}, & (x_1, x_2) \in D_2. \end{cases} \tag{41}$$

For this purpose, the right-hand sides f_{D_1}, f_{D_2} and the function $\beta(s)$ in (38)–(40) were defined from given

values of u_{D_1} and u_{D_2} by the formulas

$$f_{D_1} = L_{D_1}u_{D_1}, \quad f_{D_2} = L_{D_2}u_{D_2}, \quad \beta(s) = \frac{\partial u_{D_1}}{\partial n}\left(\frac{\partial u_{D_2}}{\partial n}\right)^{-1}. \tag{42}$$

Thus, the test problems were to approximately recover $u(x_1, x_2)$ defined by (41) from the functions f_{D_j} and $\beta(s)$ given by (42) and from the corresponding vector function φ. The normed space Φ was specified as the space $V_{\Gamma_1} = V_{\Gamma_2}$ of Cauchy data with norm (17c).

In the experiments, we monitored the relative error of the approximate solution u_{calc} found according to the algorithm. The relative error was defined by the formula

$$\varepsilon = \max_{x_1, x_2}|u_{\text{calc}} - u_{\text{exact}}|\,(\max_{x_1, x_2}|u_{\text{exact}}|)^{-1}.$$

We considered two test problems.

Problem 1. Let $c_3 = 0$. Then, by the construction of $u^{(2)}$, $\beta(s) \equiv 1$.

Problem 2. Let $c_3 = 1$. Then, $\beta(s)$ is calculated by formula (42).

The algorithm was implemented with various difference auxiliary problems used for designing $G_j(v_{\Gamma_j})$ and $\bar{u}_{D_j}$, $j = 1, 2$.

To construct $G_1(v_{\Gamma_1})$, we used the problem for the difference Poisson equation in the square $-2 \le x, y \le 2$ with zero Dirichlet conditions. The sides of the square belonged to lines of a square grid. The mesh size was characterized by the number n_1 of grid intervals lying on the square side. The number n_1 was a parameter of the algorithm.

To construct a particular solution $\bar{u}_{D_1}$, we also used the difference Dirichlet problem in the square $2 \le x, y \le 2$ with Dirichlet conditions coinciding with those in the original problem, but the mesh size was characterized by another number m_1 of grid intervals lying on the square side.

As an auxiliary problem for constructing $G_2(v_{\Gamma_2})$, we used the difference Dirichlet problem in the square domain

$$-a_2 \le x, \quad y \le a_2, \quad 1 < a_2,$$

with zero Dirichlet data on a square grid whose mesh size was characterized by the number n_2 of grid intervals lying on the square side. Here, a_2 and n_2 are parameters.

The auxiliary problem for computing the particular solution $\bar{u}_{D_2}$ was of the same form, but a_2 was replaced with an a_2-independent parameter a_{2f} and the number m_2 of grid intervals lying on the square side was independent of n_2.

Table 1

Version	N	m_1	m_2	c_1	c_2	a_2	a_{2f}	k	e	ε	ε_u
1	18	256	256	1	1	1.6	1.7	1	0.0	5.3×10^{-5}	4.6×10^{-5}
2	"	1024	1024	"	"	"	"	"	"	5.8×10^{-6}	2.9×10^{-6}
3	14	"	"	"	"	"	"	"	"	4.2×10^{-3}	"
4	18	128	"	"	16	"	"	0	"	6.1×10^{-4}	1.2×10^{-4}
5	"	"	"	"	"	"	"	1	"	1.6×10^{-4}	"
6	42	256	512	4	"	"	"	"	"	5.6×10^{-4}	5.5×10^{-4}
7	30	"	"	"	"	"	"	"	"	4.6×10^{-3}	"
8	18	128	128	1	1	1.3	1.3	"	"	1.6×10^{-4}	1.1×10^{-4}
9	30	"	"	"	"	1.6	1.7	"	0.1	3.0×10^{-4}	2.1×10^{-4}
10	22	"	"	"	"	"	"	"	"	3.7×10^{-4}	"
11	"	1024	1024	"	"	"	"	"	"	1.6×10^{-4}	3.3×10^{-6}

Table 2

N	m_1	m_2	ε
42	128	128	3.0×10^{-4}
"	256	256	7.6×10^{-5}
22	128	128	3.0×10^{-4}
"	256	256	7.4×10^{-5}
14	128	128	4.5×10^{-3}
"	256	256	4.3×10^{-3}

For $n_1 = n_2 = 512$, the numerical results obtained for Problem 1 (with $\alpha = 0.2$ and $\beta = 1$) are listed in Table 1 and those obtained for Problem 2 (with $c_1 = c_2 = 1$, $a_2 = 1.6$, $a_{2f} = 1.7$, $k = 1$, $\alpha = 0.2$, and $\beta \neq 1$) are given in Table 2.

The desired solution to Problem 1 is the classical solution to Poisson's equation in the square $-2 \le x, y \le 2$. Therefore, along with the known analytical expression for this solution, which was recovered using the algorithm, this solution was approximately calculated by applying a five-point difference scheme. The mesh size was the same as that used for finding the particular solution $\bar{u}_{D_2}$. Table 1 presents the relative error ε_u in this solution, which shows the maximum accuracy that can be reached by any five-point difference algorithm if its mesh size is no less than that used to calculate this solution.

Let us comment on the results given in the tables.

The basic result for Problem 1 is that, for a sufficient number of basis functions and for sufficiently fine grids used in four auxiliary problems, the error in the resulting approximate solutions differs little from ε_u. Consequently, these solutions cannot be improved considerably.

Furthermore, the accuracy of the results obtained for given mesh sizes is improved when the Taylor operator $\pi^{(k)}_{\Gamma_{jh}\Gamma_j}$ is calculated by the three-term Taylor formula ($k = 1$) rather than the two-term one ($k = 0$). This can be seen by comparing versions 4 and 5 in Table 1.

The number N of harmonics used in the approximate representations of $u|_{\Gamma_j}$ and $\left.\frac{\partial u}{\partial n}\right|_{\Gamma_j}$ influences the accuracy if the mesh sizes are rather small. For example, versions 6 and 7 in Table 1 show that the accuracy is reduced by roughly 50 times when the algorithm switches from 42 to 30 harmonics. An analogous saturation effect occurs in the auxiliary problems when the number of harmonics is fixed and the mesh sizes are decreased. In this case, a further refinement of the grids does not improve the accuracy.

The norm $\|v_{\Gamma_{jh}}\|_{V_{\Gamma_{jh}}}$ defined by (27) is a function of α. Nevertheless, in version 2, the accuracy of the results hardly change when $\alpha = 0.2$ is replaced by $\alpha = 0.02$ or $\alpha = 2$.

This can be explained by the low dimension of the spaces spanned by $\pi^{(1)}_{\Gamma_{jh}\Gamma}\Psi_k$ and $\pi^{(1)}_{\Gamma_{jh}\Gamma}\Psi'_k$. An idea of the number of basis vector functions that are sufficient for obtaining a fairly accurately approximation of the desired Cauchy data is given by the following formulas for the Cauchy data of the desired solution, which were designed in advance:

$$u_\Gamma = \begin{pmatrix} u|_\Gamma \\ \left.\frac{\partial u}{\partial n}\right|_\Gamma \end{pmatrix}, \quad c_1 = 1,$$

$$u_\Gamma \approx 71.57 - 1.2\cos(4\theta) + 1.9 \times 10^{-4}\cos(8\theta),$$

$$\left.\frac{\partial u}{\partial n}\right|_\Gamma \approx -98.56 - 4.6\cos(4\theta) - 1.5 \times 10^{-3}\cos(8\theta).$$

Therefore, the constants of norm equivalence on the corresponding finite-dimensional subspaces of $V_{\Gamma_{jh}}$ are close to unity. Even for $\alpha = 0$, the normal derivative $\left.\frac{\partial u}{\partial n}\right|_{\Gamma_{jh}}$ on Γ_{jh} is taken into account by using only the terms u_ν^2 ($\nu \in \Gamma_{jh}$) since the difference boundary is represented by two levels.

For $\beta \neq 1$, Table 2 suggests conclusions similar to those drawn by inspecting Table 1.

To conclude, we note the following point. Because of the rapid oscillations in the solution inside D_2, a small mesh size is required in the computation of $\bar{u}_{D_2}$ in order to achieve good accuracy. However, a small mesh size in D_1 obviously leads to high computational costs.

Our approach makes it possible to pick both the auxiliary problems and the mesh size for determining the particular solutions $\bar{u}_{D_1}$ and $\bar{u}_{D_2}$ and, then, for calculating v_{Γ_j}, and G_j can be chosen independently.

Note that the difficulty associated with the inhomogeneous behavior of the solutions could be overcome by applying an irregular grid and finite element equations. We used regular grids to cope with this difficulty.

In Problem 2, in addition to the necessity of using different mesh sizes in D_1 and D_2, we encountered difficulties associated with the discretization of the conditions on Γ, which were also overcome by using our approach.

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