Kathleen Cramer,
Debra Monson,
Stephanie Whitney,
Seth Leavitt, and
Terry Wyberg

Dividing

ISTOCKPHOTO.COM

Fractions and Problem Solving

See how a class of sixth graders used concrete and pictorial models to build meaning for arithmetic operations with fractions.

Invert and multiply. Fraction division is generally introduced in sixth or seventh grade with this rule. We examined current commercial curricula and found that few textbooks use context as a way to build meaning for the division of fractions. When context is used, the connection between the invert-and-multiply rule and the context is superficial at best.

Textbooks often use illustrations as a form of representation to build meaning. However, the transition from these pictures to the symbolic rule occurs quickly. In so doing, students may be getting an inadequate understanding of fraction division.

One way that textbooks show division with fractions is by providing a picture for a single problem, such as 3 ÷ 1/2. In this case, students would be shown three rectangles, each divided in half, so that the solution of 6 is recognized (see **fig. 1a**). The invert-and-multiply algorithm is presented after this one example. It is likely that the pictures will appear once or twice in the practice set following the example.

Another common textbook method is exemplified by the example of

$$\frac{8}{9} \div \frac{2}{9}.$$

A rectangle is divided into nine pieces, eight of which are shaded to represent 8/9 (see **fig. 1b**). The caption, or explanation, in the book would group the sections two at a time, into four portions, yielding the result. The symbolic representation and algorithm of the example follows (in **fig. 1c**). Each of these examples demonstrates a quick transition to a symbolic rule.

Our approach differs from the way that fraction division is introduced in traditional curricula. We presented fraction division as a problem-solving opportunity for students. Students constructed their own strategies that relied on pictures. Their strategies and pictures reflected their understanding of the part-whole model for fractions and their ability to name fractional parts when the unit changed. The story problems we used were based on a measurement model for division.

Most textbooks tend to use a *measurement model for division* (van de Walle 2007). Similar contexts can be used to write both measurement and partitive models. Consider the following two examples:

1. Kay has an 8-gallon bag of soil for her garden plants. If each planter uses 2/3 gallon, how many planters can she fill?

2. Kay has an 8-gallon bag of soil for her 12 planters. How much soil will be in each planter?

The first problem is an example of a *measurement* model; students are given a set measure, 2/3 gallon, and they must determine how many groups of 2/3 gallon are in the 8-gallon bag of soil. The second problem is an example of a *partitive* model, because students know the number of groups and are determining the amount in each group.

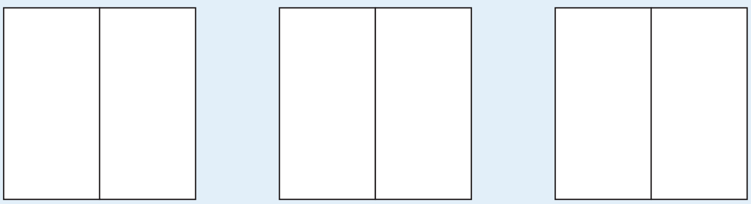
We chose to use a measurement model because we anticipated that it would give sixth-grade students an opportunity to solve fraction division problems using pictures. In so doing, this choice would support the common-denominator procedure for fraction division. Since these students had already learned to use common denominators to add and subtract fractions, the measurement model for fraction division gave them another opportunity to see how common denominators can be used when operating with fractions.

We acknowledge that this four-day problem-solving experience represents students' initial entry into division with fractions. Future instruction should include the partitive model so that it can support their construction of the invert-and-multiply procedure.

The numbers in the story problems varied; they included a whole number divided by a fraction, division of two fractions, and answers with and without remainders. The strategies that students constructed built

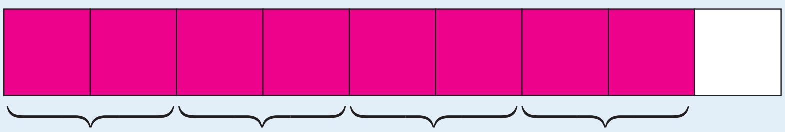
STEVE SNYDER/ISTOCKPHOTO.COM

Fig. 1 Fraction division is introduced in commercial middle school textbooks in a variety of ways.



(a)

Three wholes divided into halves is a way to represent 3 ÷ 1/2.



(b)

$\frac{8}{9} \div \frac{2}{9}$ is shown by shading 8/9 and combining into four groups.

The problem: $\frac{8}{9} \div \frac{2}{9} = 4$

Multiply by the reciprocal: $\frac{8}{9} \div \frac{2}{9} = \frac{8}{9} \times \frac{9}{2}$

Multiply and simplify: $\frac{{}^{4}\cancel{8}}{{}_{1}\cancel{9}} \times \frac{\cancel{9}^{1}}{\cancel{2}_{1}} = \frac{4}{1} = 4$

(c)

meaning for dividing fractions using a common-denominator approach.

STUDENTS' BACKGROUND

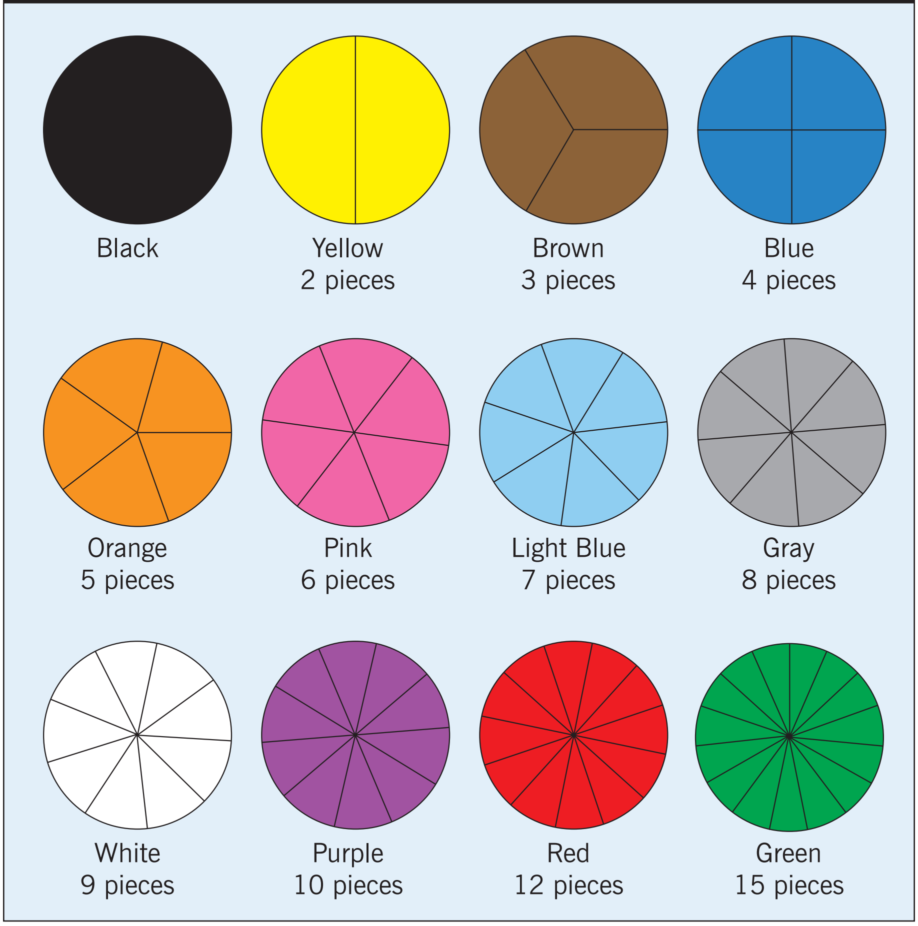
To better understand why students were able to construct their own strategies for solving division problems using fractions, we share some background on what happened before students worked with division. These sixth graders first completed a four-lesson review of fractions. They used fraction circles and paper folding to reinforce their understanding of part to whole. The lessons emphasized the importance of identifying the unit when naming fractions.

An early activity used the representations shown in **figure 2**. Students named fraction-circle pieces in many ways: the blue piece was 1/4 when the black circle was the unit and 1/2 when the yellow piece was the unit. The red piece was 1/12 when the black circle was the unit, 1/6 when the yellow piece was the unit, and 1/3 when the blue one-fourth piece was the unit. The purpose of this activity is to reinforce the part-whole construct for fractions and help students understand the important role that the unit plays in naming a fractional amount.

Students see that each piece of the circle can be named in more than one way depending on what is stated as the unit. Understanding the role of the unit is critical when using pictures to divide fractions. Readers will see how this idea plays out later in fraction division when examining student work. Reviewing this representation also supported students' construction of informal ordering ideas and reinforced the meaning of equivalence.

Students then participated in a series of twenty-eight lessons on fractions to build meaning and procedural skill when using decimals as well as addition, subtraction, multiplication, and division. Using concrete and pictorial models and real-world contexts helped build meaning. They also made sense of the symbolic procedures by making connections between the models and the steps of the algorithms. The models included fraction circles, paper folding, pictures of rectangles and circles, and the number line. The last four lessons introduced students to fraction division.

Fig. 2 Students see that each circle piece can be named in more than one way, depending on what unit is used.



DIVISION USING WHOLE NUMBERS

We launched the lessons using this measurement-division story problem with whole numbers:

> Addis is a baker. She bought 50 pounds of flour. How many 5-pound containers can she fill?

Students were asked to imagine how they would solve this problem.

We wanted students to connect this measurement problem to the question of "How many groups of 5 pounds are in 50 pounds?" and to the division sentence 50 ÷ 5 = 10. This problem set the stage for students to transfer their understanding of measurement story problems with whole numbers to situations involving fractions.

Students were asked to work in groups to solve three story problems (see **fig. 3**). Group work was followed by a class discussion in which groups' solution strategies were presented. Before reading through examples of students' solutions, solve each problem using pictures instead of the algorithm you may have used in the past.

Fig. 3 Story problems involving measurement can introduce fraction division.

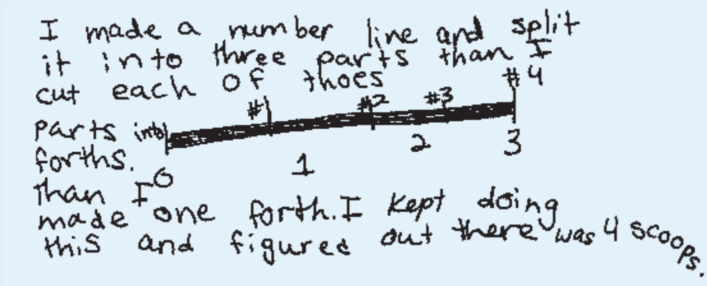
1. A scoop holds 3/4 cup. How many scoops of bird seed are needed to fill a bird feeder that holds 3 cups of bird seed? Show how to use pictures to solve this problem. Draw a picture of your solution below. Explain your solution in words.

2. You bought 4 pints of ice cream from Ben & Jerry's© for your party. You plan on serving each friend about 2/3 pint. How many servings can you dish out? Show how to use pictures to solve this problem. Draw a picture of your solution below. Explain your solution.

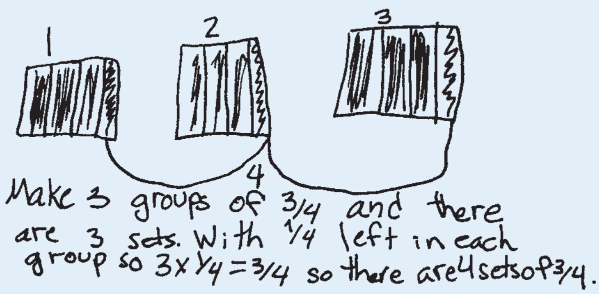
3. You have 4 cups of lemonade concentrate. If you mix 2/5 cup of the concentrate with 1 gallon of water, you can make a pitcher of lemonade. How many pitchers of lemonade can you make with those 4 cups of concentrate? Show how to use pictures to solve this problem. Draw a picture of your solution below. Explain your solution in words.

Fig. 4 Student solutions to the Bird Seed problem use a variety of pictures.

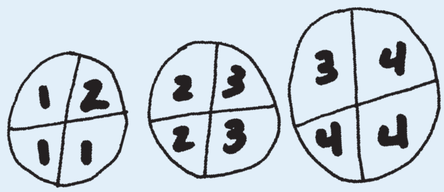
1. A scoop holds 3/4 cup. How many scoops of bird seed are needed to fill a bird feeder that holds 3 cups of bird seed? Show how to use pictures to solve this problem. Draw a picture of your solution below. Explain your solution in words.



(a)
April's solution using a number line



(b)
India's text and drawings



(c)
Xander's circular solutions

Figure 4 show three students' solutions. In (a), April constructed a double number line. Before the start of the division lessons, students used number lines to solve fraction multiplication tasks. Several students constructed double number lines when modeling fraction multiplication and transferred this approach to situations involving division.

April labeled the number line in two ways to make sense of the problem. The numbers 1, 2, and 3 that are underneath the line represent the 3 cups of bird seed. After partitioning each of the three units into fourths, April counted out a group of 3/4 and labeled it 1; she counted another 3/4 and labeled it 2; and so on. The numbers 1, 2, 3, and 4 above the line represented the number of scoops. The double-number-line model that April constructed helped her coordinate the number of cups of bird seed used and the number of scoops.

India drew 3 rectangles to represent the 3 scoops of bird seed (see **fig. 4b**). She partitioned each rectangle into fourths and shaded 3/4 of each rectangle to make "3 groups of 3/4." She combined the three remaining fourths in each rectangle to create one more scoop.

Xander drew the picture shown in **figure 4c** on an overhead-projector transparency as he shared his solution with the other students. He drew 3 circles to represent the 3 cups of bird seed that the bird feeder could hold. He partitioned each cup into fourths and counted out groups of 3/4 by naming the first 3/4 group 1, 1, 1; the second 3/4 group 2, 2, 2; and so on.

What these three solutions have in common are a picture that represents the initial amount being divided, a partitioning of the picture into fourths, and a system to keep track of

Fig. 5 Haley's unique strategy involved repeatedly drawing 1/3.

You have 3 cups of candy hearts. You plan on making little bags of hearts. Each bag will hold 1/3 cup. How many bags can you make?

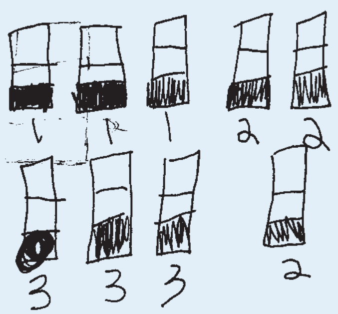
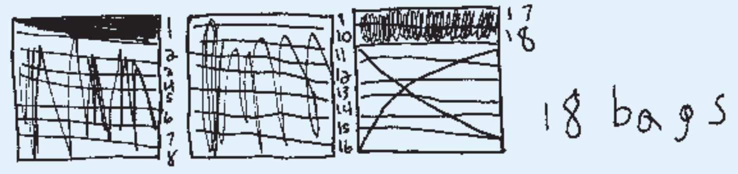


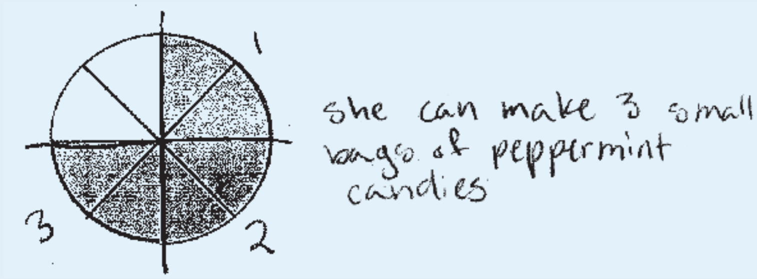
Fig. 6 Students were able to correctly arrive at an answer when an initial picture is provided.

1. Kia has 2 1/4 pounds of peppermint candies. She wants to put them in small bags, each about 1/8 pound. How many small bags of candies can she make?



(a)

2. What if Kia had only 6/8 pound of peppermint candies and she makes small bags, each about 1/4 pound. How many small bags of candies can she make?



(b)

the groups of 3/4 being removed from that amount. Each picture models a solution to the question "How many 3/4 are in 3?" and demonstrates division as repeated subtraction.

Most of the students' initial solution strategies involved drawing pictures similar to those made by April, India, and Xander. We did notice that a few students constructed a different picture strategy.

The story problem in **figure 5** was part of students' homework. Haley's solution demonstrates an alternative picture solution. She did not start with a picture of the 3 cups of candy hearts: Her picture represented 1/3 cup put into each bag. She kept drawing pictures of 1/3 until she reached 3 cups. Although she answered the question "How many 1/3 are in 3?" her thinking involved adding groups of 1/3 to reach 3 as opposed to the three previous solutions in which students took away groups of 3/4 from 3.

Students who used this counting-up strategy made errors because, in some cases, they did not know when to stop counting or when they reached the total amount. During the class presentations of students' solutions, we discussed the two different ways to draw pictures. We reached consensus that solving the division tasks as illustrated by April, India, and Xander was easier to understand. We reinforced this strategy in subsequent lessons by giving students the story problem and an initial picture to use to solve the problem.

Two sample problems in which the initial picture was provided are shown in **figure 6**. In (a), the student partitioned each whole into groups of 1/8 and shaded 2 1/4 so that she could easily count out groups of 1/8 that were in 2 1/4. In (b), the original picture given to the students was partitioned into eighths. To count out fourths, students needed to change the picture to 3/4 to see that three groups of 1/4 can be removed from 6/8. We varied the numbers in the problems so at times students needed to increase the number of partitions; at other times, students would need to eliminate some of the partitions.

Students' solutions were similar in that they looked for a way to change the picture so that each number in the fraction division task had the same denominator. Students were able to do this intuitively, without direct instruction. When solving 3 ÷ 3/4, they changed the picture to show twelve-fourths and then identified groups of three-fourths. When students solved 2 1/4 ÷ 1/8, their picture became 18/8 ÷ 1/8. And when solving 6/8 ÷ 1/4, the picture was 3/4 ÷ 1/4.

Using pictures, students constructed a common-denominator procedure for fraction division. This connection between their picture solutions using common denominators and the symbolic procedure using common denominators was brought up later in the lessons. It is significant that this alternative algorithm to invert and multiply was constructed by students themselves as they solved measurement division story problems with fractions and involved limited direct instruction from the teacher.

DIVISION WITH REMAINDERS

In each example shown so far, the answer is a whole number. How did these students make sense of fraction

Reflect and Discuss

Reflective teaching is a process of self-observation and self-evaluation. It means looking at your classroom practice, thinking about what you do and why you do it, and then evaluating whether it works. By collecting information about what goes on in our classrooms and then analyzing and evaluating this information, we identify and explore our own practices and underlying beliefs.

The following questions are suggested prompts to help you reflect on the article and on how the authors' ideas might benefit your own classroom practice. You are encouraged to reflect on the article independently as well as discuss it with your colleagues.

- After examining how students in this article used pictures to solve fraction division problems in **figure 4**, what types of pictures do your students draw to solve measurement division problems with fractions?
- How do your students use prior knowledge about fractions and division to solve contextual problems involving division of fractions?
- How do your students handle results that have a remainder? What is an effective way to help them understand the meaning of a remainder in fraction division problems?

You are invited to tell us how you used Reflect and Discuss as part of your professional development. The Editorial Panel appreciates the interest and values the views of those who take the time to send us their comments. Letters may be submitted to *Mathematics Teaching in the Middle School* at **mtms@nctm.org**. Please include "Readers Write" in the subject line. Because of space limitations, letters and rejoinders from authors beyond the 250-word limit may be subject to abridgement. Letters also are edited for style and content.

division problems with remainders?

Before examining these examples of students' thinking, work through the problem in **figure 7a** that we used to initiate our discussion on labeling the remainder. Estimate first. Do you think you will have at least 3 servings? Could you get 6 servings?

We invited students to come to the front of the class and share their drawings and explanations. Scout drew circles for the 3 pounds of fish and partitioned each circle into thirds (see **fig. 7a**). She used different colors to designate each serving. She pointed to the 1 piece left over and commented that it was 1/3 of the whole circle. She also stated and recorded that the serving size was 2/3 and renamed the amount left over as 1/2 of a serving. Her final answer was 4 1/2 servings.

Determining the unit is critical when using pictures to solve fraction division problems with remainders. Our students had many experiences naming fractions in which the unit varied (see **fig. 2**). They understood that a fraction amount represented by a concrete model or picture has meaning only if they know the unit. They realized that this amount could be named in more than one way; it all depended on the unit. Understanding that an amount can be named in more than one way is critical to making sense of fraction division with remainders.

Not all students in the class agreed right away that the answer should be named 4 1/2 servings. Some argued that it should be 4 1/3 servings. A lively discussion among class members led most students to realize that the remainder needed to be named by comparing it with the serving size and not the whole circle (1 pound of fish) that Scout drew.

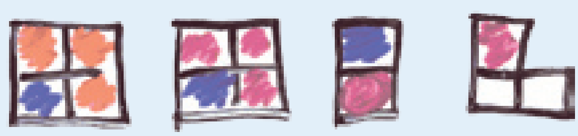
Figure 7b lists the problem and shows Ty's picture that he presented to the class. Ty used squares to represent 2 1/2 pounds. He partitioned the 2 1/2 pounds into fourths and counted out 3/4 using different colors. To explain to the rest of the class why the one part left over is named 1/3, he

Fig. 7 Students used color and counting to solve division with remainder problems.

You have 3 pounds of fish. You are planning on serving each person 2/3 pound of fish. How many full servings are possible? How can you name the amount that is left over?

(a)
Scout's colorful solution strategy

You have 2 1/2 pounds of fish, but now the serving size is 3/4 pound. How many full servings will you have? How can you describe the amount left over?



(b)
Ty's squared-off sizes

drew the serving size of three boxes to the right of the original picture. He explained that the one part left over is 1/3 of the serving size.

The next step was to guide students to translate their actions with pictures to symbols. This is a critical transition. We believed that when students were able to verbally describe the steps they took to solve the problem using pictures, they were ready to record those steps with symbols.

We asked students to reflect on their pictured solutions and to describe verbally what they did. We guided this discussion by asking them to find the original number sentence for the problems and describe how they changed the picture to solve the problem. Students then recorded the action of changing the drawing into a number sentence that had common denominators.

A story problem that can be represented symbolically with the number sentence 4 ÷ 2/3 is shown in **figure 8**. To solve the problem, the student partitioned four wholes into thirds to make it easier to count out groups of 2/3. The original number sentence represents the picture before the partitioning; the second number sentence represents the change in the picture that makes it possible to count out groups of 2/3.

The students' number sentences became records of what they could do with pictures and what they could verbalize. At this point, we did not ask students to generalize the common-denominator algorithm, although some students did observe that you could just divide the numerators once you rewrote the sentence with common denominators. They reasoned that to evaluate 12/4 ÷ 3/4, they have to determine the number of groups of 3 pieces (of size 1/4) that are in a set of 12 pieces of the same size. As this was only a series of four lessons, we did not set as our goal the generalization of the common-denominator algorithm with symbols. For most of these sixth-grade students, solving fraction division still involved first drawing pictures and then attaching the number sentence as an added connection.

STUDENT ERRORS

What were some common errors that students made as they worked through fraction division? We describe two examples, which are shown in **figure 9a** and **9b**.

One error involved Joey's work (see **fig. 9a**). He knew that the answer was 3 when looking at the picture. However, when he recorded the number sentence, he wrote 3/4. Because the problem involved fractions, many students, including Joey, may have erroneously felt that the answer should be a fraction even though the picture represented a whole-number solution.

It is also possible that these students may have been taught to label their answers; therefore, they would have written 3/4 instead of 3 because they viewed the fourths as the "label." To overcome this error, we asked students to state aloud the answer to the question embedded in the story problem. For example, students explained the answer to the question "How many 3/4 are in 3?" by saying, "I took out 4 groups of 3/4 in 3" or "There are 4 groups of 3/4 in 3."

Fig. 8 In this example, the student was able to make connections between the picture and the number sentence using a common denominator.

You have a piece of lumber that is 4 yards long. You want to cut lengths 2/3 yard long to use in a shelf you are building. How many shelves can you cut?

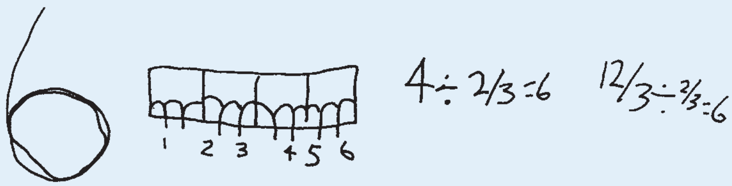
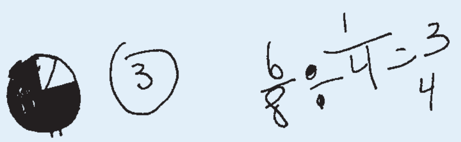


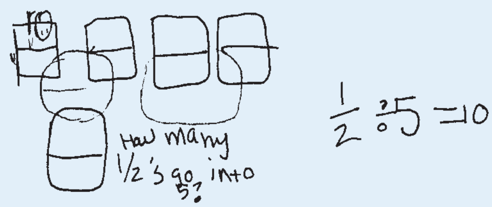
Fig. 9 Some common errors made by students when dividing fractions may involve labeling and transpositions.

Draw a picture of the fraction 6/8. How many 1/4 are in that amount?

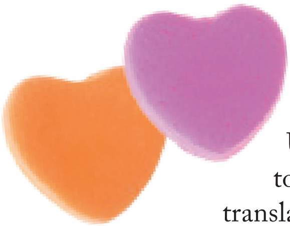


(a)
Joey incorrectly wrote 3/4.

A scoop holds 1/2 cup. How many scoops of bird seed are needed to fill a bird feeder that holds 5 cups of seed?



(b)
Hazel's error promoted class discussion.



Using language to facilitate the translation from the picture to symbols helped students overcome this error.

Another error involved recording the division sentence for the story problem (see **fig. 9b**). In the first few fraction division lessons, we asked students to record the division sentence for the story problem. We did not ask them to rewrite the number sentence with common denominators. Some students were unsure how to write the division sentence. Hazel could solve the problem with a picture and write a sentence to describe what she did but did not know how to record these actions using a division number sentence. She wrote $1/2 \div 5 = 10$ instead of $5 \div 1/2 = 10$. This error created an opportunity for a class discussion on how to write division sentences.

SUMMARY

The part-whole model for fractions helps students understand the use of rational numbers (English and Halford 1995). Although it is important for middle school students to expand their interpretation of fractions to other constructs (operator, ratio, quotient, and measurement), the part-whole model is a crucial gateway to making sense of rational numbers (Lamon 2007). These sixth-grade students' extended use of concrete and pictorial models provided them with a deep understanding of the part-whole model that in turn supported their construction of a procedure for dividing fractions. This initial fraction interpretation supported students' understanding of a more complex idea: how to divide fractions.

Students' understanding of the part-whole model went beyond naming and representing pictures and concrete models using fraction symbols. Our students' understanding of the part-whole construct was strongly tied to the idea of "flexibility of unit." Early in their exposure to naming and representing fractions, students were asked to reflect on the unit. The idea that a certain piece of a fraction circle could be named in multiple ways depending on the unit is a fundamental idea tied to the part-whole model. Our students have many experiences reinforcing this idea; this is not the case in traditional curricula.

Our students' understanding of fraction division is closely tied to their actions with pictures. Helping students make the connection between symbolic procedures and the actions they perform with concrete or pictorial models can enhance their understanding of a procedure (English and Halford 1995). In this case, meaning for the symbolic procedure is developed from students' initial actions with pictorial models. There is a strong connection between actions on the pictures and steps to the common denominator algorithm. Students constructed the common-denominator strategy for fraction division by solving a measurement story with pictures, with some guidance from the teacher. Instructional activities linked their pictorial strategy to an abstract one.

We should not underestimate what students are capable of doing in mathematics. As we have seen, this group of sixth graders were given the opportunity to develop a rich understanding of the part-whole model for fractions. They were able to construct for themselves a meaningful interpretation for fraction division on which future symbolic work can be built.

In future lessons, students should be given story problems involving partitive division. These types of problems and experiences can be used to help students develop meaning for the invert-and-multiply algorithm. A more complete understanding of fraction division would involve working with measurement and partitive contexts as well as constructing more than one symbolic procedure.

REFERENCES

English, Lyn D., and Graeme S. Halford. *Mathematics Education: Models and Processes.* Mahwah, NJ: Lawrence Erlbaum Associates, 1995.

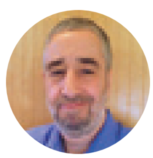
Lamon, Susan. "Rational Numbers and Proportional Reasoning: Toward a Theoretical Framework for Research." In *Handbook of Research on Mathematics Teaching and Learning*, edited by Douglas Grouws, pp. 629–67. Reston, VA: National Council of Teachers of Mathematics, 2007.

Van de Walle, John A. *Elementary and Middle School Mathematics: Teaching Developmentally.* Boston, MA: Pearson Education, 2007.

This material is based on work supported by the National Science Foundation under Grant Number 0628005.







Kathleen Cramer, crame013@umn.edu, and **Terry Wyberg**, wyber001@umn.edu, are colleagues at the University of Minnesota. Their research interests focus on teaching and learning rational numbers. **Debra Monson**, Debbie.monson@comcast.net, and **Stephanie Whitney**, whit0818@umn.edu, are doctoral candidates in mathematics education at the University of Minnesota working on the Rational Number Project. **Seth Leavitt**, Seth.Leavitt@mpls.k12.mn.us, is a teacher at Field Middle School in Minneapolis. He is interested in improving mathematics instruction in kindergarten through high school classrooms.