Homework 1

Due: Friday, 1/16/2004

1. Let G be a group. Prove that $H \subset G$ is a subgroup if and only if $h_1 h_2^{-1} \in H, \quad \forall h_1, h_2 \in H$.

2. Let $f : G \to H$ be a homomorphism of two groups. Prove the following

   (a) $Kerf \lhd G$ and $Imf < H$.

   (b) $G/Kerf \cong Imf$ – aka Isomorphism Theorem

3. Let $\gamma : G \to Aut(G)$ be a homomorphism given by $\gamma(g) = f_g$, where f_g is conjugation by $g \in G$. Show that $Ker\gamma$ is the center of G (subgroup of G consisting of all elements that commute with every other element of G). Also show that $Im\gamma = Inn(G) \lhd Aut(G)$.

4. Show that each equivalence class $[w]$ of words in alphabet X contains exactly one reduced word. Hint: assume there are two and that you have a sequence of words that lead from one to the other (by inserting and deleting subwords of the form xx^{-1}) so that the sum of the lengths of those words is minimal possible.

5. Using Tietze transformations show that the cyclic group $\mathbb{Z}_6 =< a|a^6 >$ is the direct product of $\mathbb{Z}_2$ and $\mathbb{Z}_3$, $< b, c|b^2, c^3, [b, c] >$. Explain how this presentation relates to the usual description of the direct product of $\mathbb{Z}_2$ and $\mathbb{Z}_3$.

6. If G acts on X and Y show that $g(x, y) = (gx, gy)$ defines an action of G on $X \times Y$. We call this diagonal action of G on $X \times Y$. What is the stabilizer of a point $(x, y) \in X \times Y$? If G acts transitively on both X and Y, is the diagonal action necessarily transitive?