Journey into Geometries

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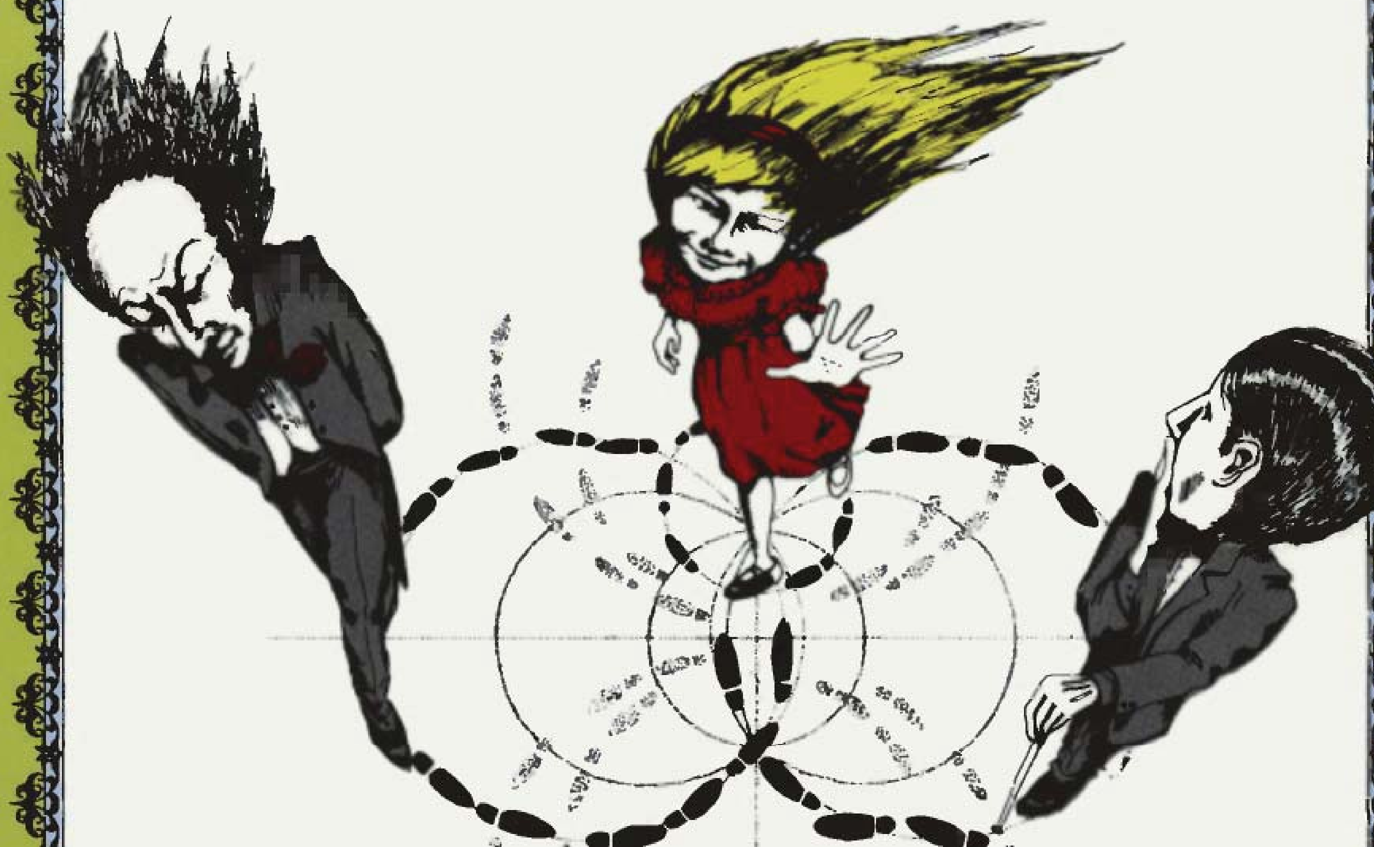
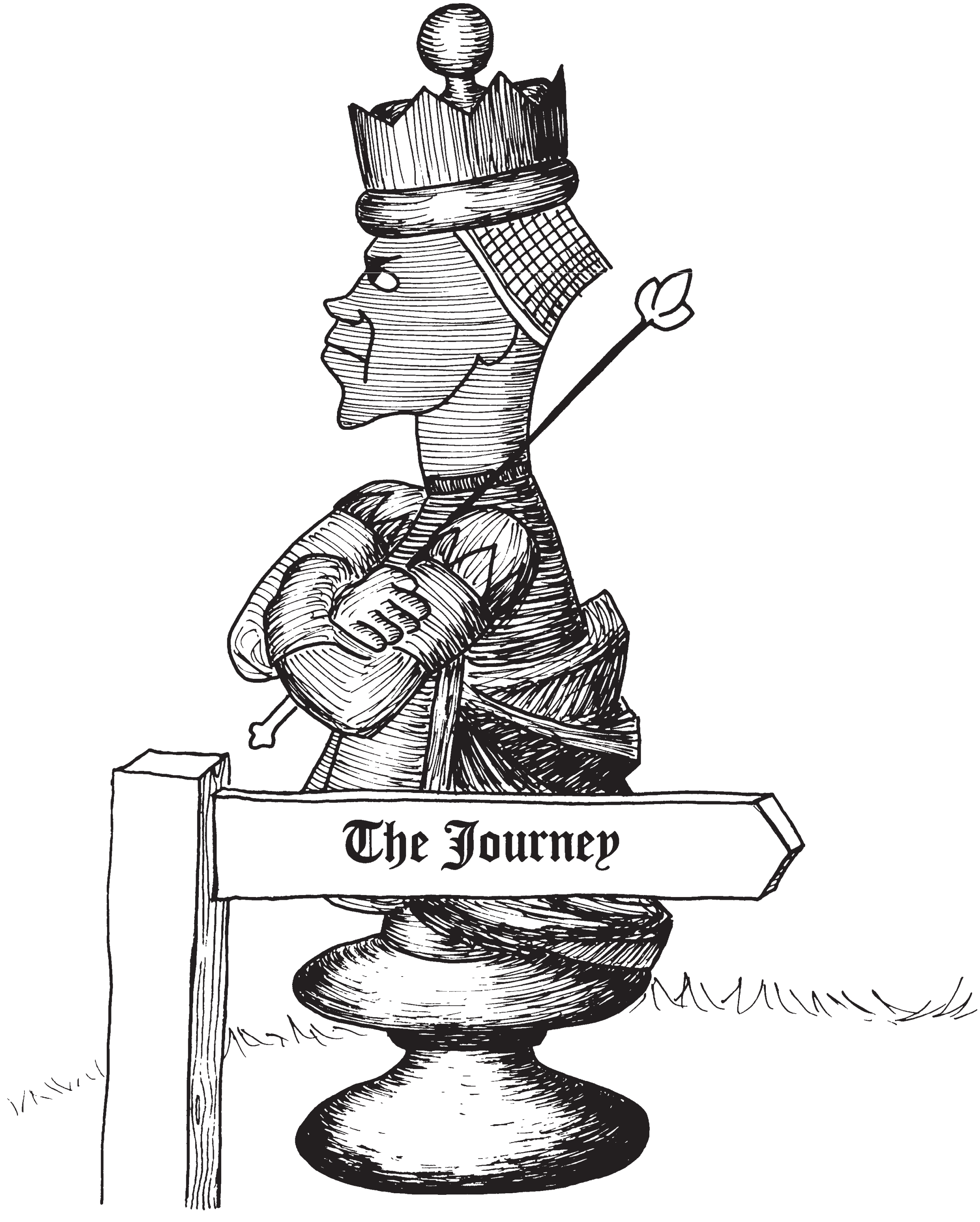


The Mathematical Association of America

Part I

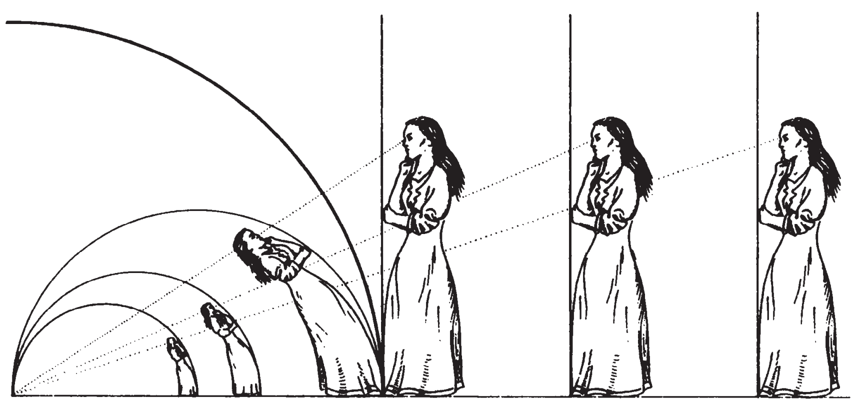


Introduction

ALICE was standing in front of her mirror. It was mid-afternoon on a summer day and she felt hot and tired in her tightly laced stays.

"Since the time I left school," she reflected, reflected ... Alice heard herself speaking, or did she hear strange voices? "Of course, reflecting is just the thing to do when you are in front of a mirror. Oh, Alice, be careful, those days through the looking glass are long past. I was a child then, now I am past finishing school."

"Hold yourself straight Alice, straight, straight, and walk away from that glass!" It was the voice of Miss Prim, that strict governess, or was it the Red Queen?



"Oh, oh, I am bending backwards! And in a perfect circular arc, too. I could never do it before! My head will soon reach the ground."

"Your head will never reach the ground, you poor unliberated Victorian creature," boomed the voice. No, it was not Miss Prim. It was not the Red Queen either.

"Of course, you've never heard of *inversion*."

"Inversion??"

"Of course, you have not heard of it. You hardly know *any* geometry!"

"But I do know my geometry. I do! I do! I know my Euclid. I know all the *postures*."

"Say *that* to Miss Prim. You mean postulates!"

"Oh, yes, postulates, and prepositions too!"

"Do you know anything about non-Euclid? Is your geometry liberated?"

Alice leaned against the French window. She opened her eyes, breathing hard. "I must see Uncle Lewis Carroll. I am so confused."

A few minutes later she was in Lewis Carroll's study. It was dark and cool and lined with volumes and volumes of mathematics. She tried to recall her thoughts or her dream or whatever it was.

"You are a deep sinker, child," said Lewis Carroll.

"You mean, deep thinker."

"I mean, deep sinker. What about that time when you went down the rabbit hole? You nearly fell through the Earth! Why, had I not saved you, you would have gone all the way to meet the *antipathies.* (That's what you called them.) But, my dear, you could not have stopped there."

"Why not?"

"You would have fallen *back.*

The rabbit hole's not hard to enter,
But it runs right through the centre,
Through hemispheres South and North
Alice sallies back and forth,
Like a pendulum, like a swing
Like a tuning fork, like a spring;
Her predicament is chronic,
She moves in a *simple harmonic.*"

"Oh, Uncle Lewis Carroll, I wish you talked *sense* just for once! I was not speaking about the rabbit hole. I was speaking about the looking glass and about *geometry.*"

"*Geometries*"

This was another voice. Alice noticed another figure amongst the dark shadows. Was it the Mad Hatter without his hat?

"Alice, have you met Dr. Whatif?" asked Lewis Carroll.

"Pleased to meet you, Dr. Whatif. You have an unusual name, Sir. Do you come from Russia or thereabouts?" Alice sounded bewildered.

"I prefer to travel in *time.* As a matter of fact, I have come on a trip from the twentieth century. I enjoy such *incursions.*"

"You mean, excursions," Lewis Carroll tried to correct him.

"Incursions. I feel that all of you here need a little mind broadening. Leaving other things aside, take, for example, geometries ..."

"Geometries!" called Alice. "Why, Dr. Whatif, you remind me of a conversation I had with the Red Queen. I met her when I went through the looking glass. It was very long ago."

"The Red Queen?" This time Dr. Whatif was puzzled.

"She told me that they have a lot of Tuesdays, all at once. You, in the twentieth century have a lot of geometries all at once. Is it because you are all red?"

"Neither all red, nor all royalty. But we have several geometries. Speaking truthfully, they are not all of our making. You people in the nineteenth century started their development. Take Bolyai."

"And Lobachevsky, Gauss, and Riemann, and all those other fellows on the Continent," continued Lewis Carroll. "They fail to convince me. Though no flaw is visible in the arguments of the modern rivals of Euclid, the Truth of nature sides with Euclid."

Dr. Whatif smiled, turned to Alice, and asked, "Young lady, how is your Greek?"

"Greek? Why, we girls here in Oxford ...."

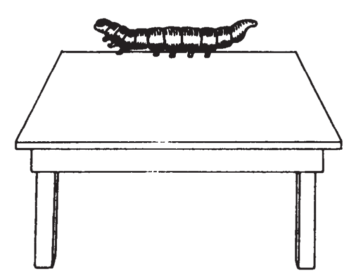
"Are not sufficiently liberated to have all your liberated hours filled with Greek and Latin. So do you even know the meaning of the word 'geometry'?"

"Yes, I do," said Alice. "'Measuring the earth', isn't it?"

"Splendid, splendid, and how do you measure the earth?"

"Why, with measuring rods."

"Like the one here." A measuring rod appeared suddenly in Dr. Whatif's hand. "Will you measure the edge of the table?"



"Why, this is not a measuring rod, this is a caterpillar!" exclaimed Alice with horror as she tried to move the rod along the table. She threw it down abruptly.

"Don't talk nonsense, my dear young lady," said Lewis Carroll. "It is just a ruler!"

"It *was* a caterpillar when I moved it. It shrank."

"But it is a *rigid* ruler. Can't you see it for yourself?"

"Are you sure that she talks nonsense? She may have *extraordinarily* fine eyesight! Maybe the ruler *does* shrink!" said Dr. Whatif.

"It is rigid!" countered Lewis Carroll.

For a moment the two dons looked at each other, then both called out at once,

"Yes, it *can* change."

"No, it can't change when you move it."

"I challenge you, can you prove it?"

Excited, they went on chanting, "Can you prove it, when you move it? Can you prove it, when you move it? ..."

Bewildered, Alice looked from one to the other. They stopped abruptly.

Whatif turned to Alice. "All right. Let us *pretend* that this rod is *not* a caterpillar. I give you as many days as you like for measuring out with it a length of 1000 miles in a *straight line*."

Quickly, Alice began to move the rod along the table, keeping her *extraordinarily* sharp eyes sufficiently far from it so as not to see the contraction of the caterpillar. She saw soon that she would not get very far moving it on the table.

"Use the floor, Alice. Then go out with it to the garden."

Alice looked at the two of them and saw that this time both were grinning in agreement. She suddenly saw the trick. "I cannot do it! There is *no* straight line on the Earth 1000 miles long. You both *know* the earth is round."

Dr. Whatif laughed. "So much for your plane geometry! 'Measuring the earth,' by moving your rod which is *really* a caterpillar in a plane which exists only in your *mind.* How can you say, my friend Lewis Carroll, that the geometry of Euclid is truth, but those of Bolyai, Lobachevsky, and the other continental fellows lead to nonsense?"

More perplexed than ever, Alice turned to her old friend, Lewis Carroll. "Please, I am not a child any more. Will you tell me what *is* that talk about Bolyai and the rest of them?"

"All right, Alice. Do you know what parallel lines are?" asked Dr. Whatif.

"Of course I do. They are lines in the plane which never meet."

"Now, then. You have also learned the fifth postulate of Euclid?"

Alice drew a breath and started reciting, "If a straight line falling on two straight lines . . . ."

"I see that you know it. Putting it simply, the postulate states that through any point not lying on a given straight line we can draw *exactly one* line parallel to the given line."

"It is *obvious* to me," cried out Alice.

"Well, to some people it was not obvious and so they tried to prove it, but no one succeeded," remarked Lewis Carroll.

Dr. Whatif continued, "So it occurred to those fellows we were speaking about that they could make different assumptions: *either*

(1) through any point not lying on a given straight line we can draw *no* line parallel to the given line, *or*

(2) through any point not lying on a given straight line there are at least two distinct lines parallel to the given line.

A geometry, different from that of Euclid, can be based on each of these two postulates."

Lewis Carroll called out, "Yes, but it's all purely fiction!"

"Still, there is no contradiction," countered Dr. Whatif.

"They will break out again in song," thought Alice with alarm. However, to her great relief Dr. Whatif continued in prose and he spoke quite seriously.

"My mathematician friends in the twentieth century are more modest in their aims than you are. They have given up chasing the 'truth.' They do not even try to define the simplest terms that we are speaking about. They know the definition game would never end."

"But I have learned *all* the definitions of Euclid," said Alice eagerly and she immediately began to recite,

"1. A *point* is that which has no part.
2. A *line* is length without breadth.
3. The extremities of a line are points.
4. A *straight line* is a line which lies evenly with the points on itself . . . ."

She wanted to continue, but Dr. Whatif stopped her.

"Why don't you tell me then what the words 'part,' 'length,' 'breadth' mean and why don't you explain how one should 'lie evenly with the points on itself'?"

Alice looked aghast.

Dr. Whatif continued, "So you see, my friends in the twentieth century occupy themselves with *'what if'* mathematics."

"Oh, you are so famous that they named their mathematics after you?"

"Well, it is the other way round. They named me after their mathematics. We like to do things the other way round. Instead of things which are thought to exist, but whose properties we do not know for certain, we prefer to speak about things of which the properties are certain, not worrying whether they exist."

"Exist? What does it mean anyway?" asked Alice.

"A good question! So we speak about points and lines and make up statements, called *axioms*, just like Euclid, such as these:

1. Through any two distinct points there is exactly one line.
2. Every line has at least two distinct points.
3. Not all points are on one line."

"What then are the *lines* and *points*?" asked Alice.

"A good question. We leave it at that."

"But if you do not even know what the points and lines are, how do you know that your statements are true?"

"We do not care. The axioms are 'if' statements. The geometry that follows from them is 'what if' geometry."

"All right," said Alice, "suppose that we have for *points*:

chocolate
peppermint
nuts

and for *lines*:

peppermint chocolate
peppermint nuts
chocolate nuts.

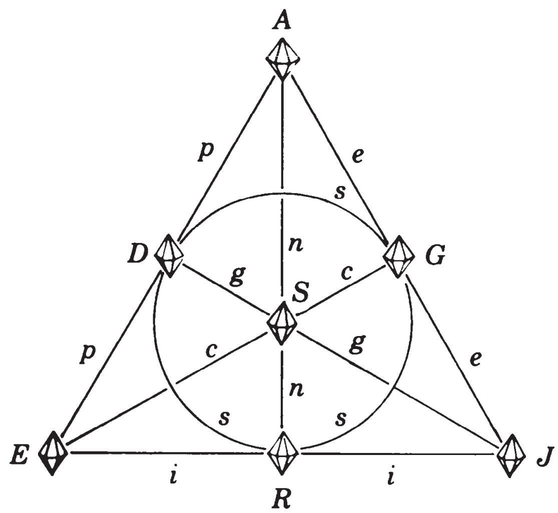
Is this a geometry?"

"Yes, a very tasty one. All axioms are satisfied."

"Tasty, but not a very *rich* geometry," countered Lewis Carroll. "Not too many points and lines in it."

"We can alter the axioms to *force* in a few more points. Say that we keep axioms 1 and 3 and alter axiom 2 so that we have: *2. There are exactly 3 points on each line.*
Add for good measure:
4. There exists one line.
(So that we know we are speaking about *something.*) And, finally:
5. There is at least one point on any two distinct lines.
Out of all this we can construct a *gemmetry* and present it to Alice."

"A gemmetry?" asked Alice.



"Yes. Our points will be gemstones and our lines will be chains containing the gemstones. It will be like this: the points will be amethyst (A), diamond (D), emerald (E), garnet (G), jade (J), ruby (R), sapphire (S). The chains will be made out of copper (c), enamel (e), gold (g), ivory (i), nickel (n), platinum (p), and silver (s), each containing exactly three gemstones. What is more, you can prove, if you wish, that with only these axioms we can make up this sort of seven point geomtry."

"But," said Alice, "couldn't we *pretend* that the *chains* are the *points* and the *gems* are the *lines*?"

"That is the *dual* geometry, I give you your due," said Dr. Whatif, "but it is still a seven point Fano geometry with the same structure."

Lewis Carroll was losing his patience,

"When everything is said and done,
Fano's only finite fun."

"You would be surprised, how many interesting things you can do with it," answered Dr. Whatif.

"Still," said Alice, "this is nothing like the good geometry we had at school with so many circles and triangles and squares and hypotenuses and problems and theorems and proofs. I just *loved* it."

"I only wanted to illustrate how axioms can be built up into a geometry. I do not deny that the Euclidean plane is *rich*, but with different axioms you can make *other* rich and varied geometries, and the whole lot together is more varied and exciting than you ever *dreamt* of."

"One moment," said Lewis Carroll, "you surely cannot just make up sets of axioms and then hope for the best!"

"Certainly not," said Dr. Whatif and turning to Alice he added,

> "With axioms, my dear, you need a gentle touch,
> They should not say too little, they should not say too much,
> And on one point above all, we have to be insistent,
> Though axioms need not be 'true,' their *set* must be consistent."

"Very neat," said Alice, "but can you make it clearer?"

"The most important thing is *consistency*. There should not be any contradiction in the geometrical theorems we deduce from the axioms. The other things are desirable, too. The axioms should be independent. If one of them can be deduced from the others, then it is not an axiom any more, but a theorem. If the axioms say too little, like the three axioms from which you obtained your sweet and tasty geometry, then there are too many different geometries which satisfy the requirements. For example, the Fano geometry, the Euclidean geometry, and the Bolyai-Lobachevsky geometries *all* satisfy those axioms."

"How can you make sure that the axioms are consistent, that there is *no* contradiction?" asked Alice.

"It is difficult to be sure. With Euclidean geometry we have many centuries of experience. In the 20th century, its foundations have been tidied up a little and we find no contradictions in it. Moreover, we all feel at home in it, don't we?"

"Yes, we do," said Lewis Carroll and Alice at once.

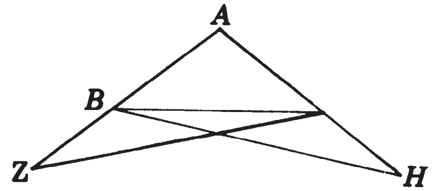
"So, when we want to test some *non-Euclidean* axioms, we make use of Euclidean geometry to test them."

"How can that be done?" asked Alice.

"I can't tell you everything at once. I can only ask, are you willing to take a journey with me? We will make an exploring trip into the inversive plane. It is really still Euclidean geometry. Then we can proceed further. Alice, is your geometry really good?"

Alice answered,

> "I know all the definitions,
> Postulates and propositions,
> And with maximum decorum
> Crossed the great 'pons asinorum'."



Here, Dr. Whatif looked a little puzzled. Lewis Carroll was quick to explain, "Pons asinorum is the 'bridge of the asses.' We gave this name to the theorem of Euclid which states that the base-angles of an isosceles triangle are equal. Euclid used a rather involved diagram to prove the theorem. It looks a little bit like a bridge, and students who could not recall Euclid's proof were silly asses."

"Once you struggled through the bridge, the road was still not easy," added Alice.

She continued, "I had a little conversation with the Red Queen when I met her. She asked me to show her the road to geometry.

'To geometry, madam, there is no royal road,' said I.

'We are not amused,' answered the Queen."

"I hope," said Dr. Whatif, "that *you* will be amused as we make our journey to geometry-wonderland."