

# Euclid and Hilbert

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## 1 Euclid

### 1.1 Axioms

1. Things which are equal to the same thing are also equal to one another.
2. If equals be added to equals, the wholes are equal.
3. If equals be subtracted from equals, the remainders are equal.
4. Things which coincide with one another are equal to one another.
5. The whole is greater than the part.

### 1.2 Postulates

1. To draw a straight line from any point to any point.
2. To produce a finite straight line continuously in a straight line.
3. To describe a circle with any center and distance.
4. That all right angles are equal to one another.
5. That if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the straight lines, if produced indefinitely, will meet on that side on which the angles are less than two right angles.

## 2 Hilbert

Remember Hilbert: "One must be able to say at all times—instead of points, straight lines, and planes—tables, chairs, and beer mugs."

We have preconceived notions of what those words mean and what the objects look like. However, from the standpoint of the theoretical development, those are undefined terms. We need to give them meaning, and we need to find out what properties and relationships they have.

### 2.1 Incidence

We start with axioms that describe the relationships between points and lines. We have three, and these are the only statements we may take for granted.

**I-1:** For any two distinct points there exists a unique line that passes through both of them.

**I-2:** For any line there exist at least two distinct points incident with it.

**I-3:** There exist three distinct points with the property that no line is incident with all three of them.

### 2.2 Betweenness

Next few axioms tell us a little more about what lines 'look' like. Pay attention however if straightness is apparent in this description. Make sure you understand what each axiom is telling you.

**B-1** If  $A * B * C$ , then  $A, B$  and  $C$  are three distinct points all lying on the same line, and  $C * B * A$  ( $B$  is between  $A$  and  $C$ ).

**B-2** Given any two distinct points  $B$  and  $D$ , there exist points  $A, C$ , and  $E$  lying on  $\overleftrightarrow{BD}$  such that  $A * B * D$  and  $B * C * D$ , and  $B * D * E$ .

**B-3** If  $A, B$ , and  $C$  are three distinct points lying on the same line, then one and only one of the points is between the other two.

**Definition 2.1.** Let  $\ell$  be any line,  $A$  and  $B$  any two points that do not lie on  $\ell$ . If  $A = B$  or if segment  $AB$  contains no point lying on  $\ell$ , we say that  $A$  and  $B$  are *on the same side of  $\ell$* , whereas if  $A \neq B$  and segment  $AB$  does intersect  $\ell$ , we say that  $A$  and  $B$  are *on opposite sides of  $\ell$* .

**B-4** For every line  $\ell$  and for any three points  $A, B$ , and  $C$  not lying on  $\ell$ :

- (i) If  $A$  and  $B$  are on the same side of  $\ell$  and  $B$  and  $C$  are on the same side of  $\ell$ , then  $A$  and  $C$  are on the same side of  $\ell$ .
- (ii) If  $A$  and  $B$  are on opposite sides of  $\ell$  and  $B$  and  $C$  are on opposite sides of  $\ell$ , then  $A$  and  $C$  are on the same side of  $\ell$ .

## 2.3 Congruence

There really are two terms that are fairly distinct that get packaged under our last undefined term. We talk about congruent segments and congruent angles. Further, the congruency most often used in high school geometry is that of triangles, but that term can be defined:

**Definition 2.2.** Two triangles are congruent if a one-to-one correspondence exists between their vertices so that the corresponding sides and corresponding angles are congruent. When we write  $\triangle ABC \cong \triangle DEF$ , we understand that  $A$  corresponds to  $D$ ,  $B$  to  $E$ , and  $C$  to  $F$ .

**C-1** If  $A$  and  $B$  are distinct points and if  $A'$  is any point, then for each ray  $r$  emanating from  $A'$  there is a unique point  $B'$  on  $r$  such that  $B' \neq A'$  and  $AB \cong A'B'$ .

**C-2** If  $AB \cong CD$  and  $AB \cong EF$ , then  $CD \cong EF$ . Moreover, every segment is congruent to itself.

**C-3** If  $A \star B \star C$ ,  $A' \star B' \star C'$ ,  $AB \cong A'B'$  and  $BC \cong B'C'$ , then  $AC \cong A'C'$ .

**C-4** Given any angle  $\sphericalangle BAC$ , and given any ray  $\overrightarrow{A'B'}$  emanating from a point  $A'$ , there is a unique ray  $\overrightarrow{A'C'}$  on a given side of the line  $\overleftrightarrow{A'B'}$  such that  $\sphericalangle B'A'C' \cong \sphericalangle BAC$ .

**C-5** If  $\sphericalangle A \cong \sphericalangle B$  and  $\sphericalangle A \cong \sphericalangle C$ , then  $\sphericalangle B \cong \sphericalangle C$ . Moreover, every angle is congruent to itself.

**C-6 (SAS)** If two sides and the included angle of one triangle are congruent respectively to two sides and the included angle of another triangle, then the two triangles are congruent.

## 3 Parallelism axioms

**Euclidean PP** For every line  $l$  and every point  $P$  not on  $l$  there exists a unique line  $m$  passing through  $P$  parallel to  $l$ .

**Elliptic PP** For every line  $l$  and every point  $P$  not on  $l$  there is no line passing through  $P$  parallel to  $l$ .

**Hyperbolic PP** For every line  $l$  and every point  $P$  not on  $l$  there exist at least two distinct lines passing through  $P$  parallel to  $l$ .

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