Math 5270: Transformational Geometry

Solutions Homework 1

July 3, 2013

1. We know the equation for a line containing point (h, k) with slope a is:

$$y - k = a(x - h)$$

In this case, we know the slope from $P_1 = (x_1, y_2)$ to $P_2 = (x_2, y_2)$ to be $a = \frac{y_2 - y_1}{x_2 - x_1}$, so we have the equation:

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

Which can be rewritten as:

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

Since any point $P = (x, y)$ on this line will satisfy the first equation, it must also satisfy the equivalent second equation.

Of course, if $x_1 = x_2$ then our equation is undefined and we have the vertical line expressed as $x = x_1 = x_2$.

In order for two lines $y = ax + c$ and $y = a_1 x + c_1$ to have a common point, it must be true that

$$ax + c = a_1 x + c_1 \Rightarrow (a_1 - a)x = c - c_1 \Rightarrow x = \frac{c - c_1}{a_1 - a}$$

This is our x-coordinate of the common point (of course the y-coordinate can be found by substituting this into either equation). However, if $a = a_1$ our equation for the x-coordinate becomes undefined, meaning that the equation $(a_1 - a)x = c - c_1$ has no solutions (when $c \neq c_1$) or infinitely many (when $c = c_1$).

Therefore, we conclude that the parallel to line l is the unique line through P with the same slope as l. We have shown that parallel lines have the same slopes and we can show there is only one such line through P by contradiction.

2. The set of all points equidistant to two points $A = (x_A, y_A)$ and $B = (x_B, y_B)$ is the perpendicular bisector of A and B. we can prove this at least two different ways:

   (a) Show that all points on the perpendicular bisector are equidistant to A and B, then show that any two points equidistant to A and B are on the perpendicular bisector. Both can be done using the Pythagorean Theorem (distance formula).

   (b) A slicker way is to use algebra. in order for point $P = (x, y)$ to be equidistant to A and B, it must satisfy the equation:

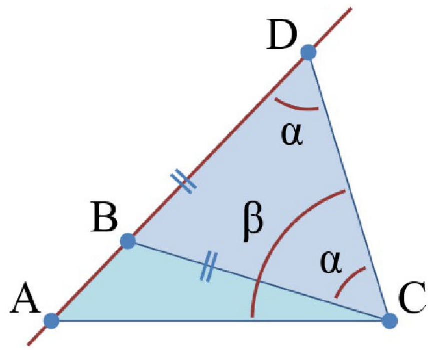
$$\sqrt{(x - x_A)^2 + (y - y_A)^2} = \sqrt{(x - x_B)^2 + (y - y_B)^2}$$

   Of course, the algebra is harder. But in the end we get the equation of a line through the midpoint of A and B with a slope that is perpendicular to the slope between A and B:

$$y - \left(\frac{y_A + y_B}{2}\right) = -\left(\frac{x_A - x_B}{y_A - y_B}\right)\left(x - \left(\frac{x_A + x_B}{2}\right)\right)$$

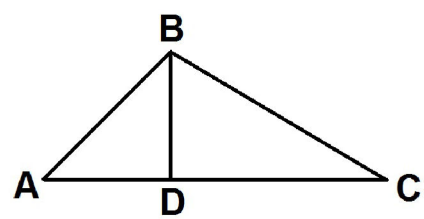
3. Again, there are several different methods:

   (a) After the style of Euclid: Given $\triangle ABC$, extend AB so that BD is the same length as BC, creating an isosceles triangle.



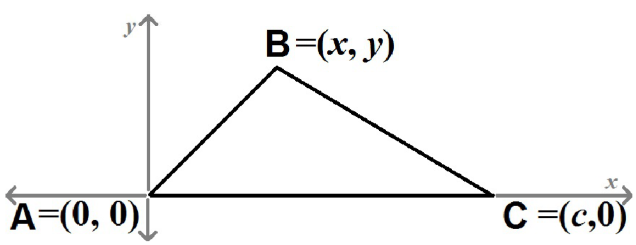
   We now have $AB + BD = AD$. Next, we must rely on the fact that larger angles of a triangle open up to larger sides. Since $\alpha < \beta$, and α opens up to AC while β opens up to $AD = AB+BD$, we have that $AC < AD = AB + BD$

   (b) Using altitudes: Given $\triangle ABC$, construct altitude BD.



   Since AD is the shortest distance from A to BD we have $AD < AB$. Likewise $DC < BC$. Therefore, $AC = AD + DC < AB + BC$.

   (c) Using algebra: Given $\triangle ABC$, place your coordinate system with origin at A and x-axis along AC such that the coordinates for A, B, and C are $A = (0, 0)$, $B = (x, y)$, and $C = (c, 0)$.



   We now have $d(A, C) = c$, $d(A, B) = \sqrt{x^2 + y^2}$, and $d(B, C) = \sqrt{(c - x)^2 + y^2}$ and want to show that $d(A, C) + d(B, C) > d(A, B)$ which is equivalent to showing $(d(A, C) + d(B, C))^2 > (d(A, B))^2$ which is equivalent to showing $(d(A, C) + d(B, C))^2 - (d(A, B))^2 > 0$ so we write out:

$$c^2 + 2c\sqrt{(c - x)^2 + y^2} + (c - x)^2 + y^2 - x^2 - y^2$$
$$= 2c\sqrt{(c - x)^2 + y^2} + 2c^2 - 2cx$$
$$= 2c(\sqrt{(c - x)^2 + y^2} - (x - c))$$

   which is certainly greater than 0 if $y > 0$. If $y = 0$ then we didn't have a triangle in the first place. Every other triangle can be placed in this position by a translation and a rotation, and those preserve distances, so triangle inequality holds for every triangle.

4. Hilbert's goal was to fix the axiomatization begun by Euclid. Consider the following intro to a great paper[1] on the matter written by Bjørn Jahren:

> Euclid's "Elements" introduced the axiomatic method in geometry, and for more than 2000 years this was the main textbook for students of geometry. But the 19th century brought about a revolution both in the understanding of geometry and of logic and axiomatic method, and it became more and more clear that Euclid's system was incomplete and could not stand up to the modern standards of rigor. The most famous attempt to rectify this was by the great German mathematician David Hilbert, who published a new system of axioms in his book "Grundlagen der Geometrie" in 1898.

   Turns out, there were a few axioms that Euclid missed ...one of these being the axiom about the number of ways lines and circles play together in a plane. There is nothing beyond our intuition of pure Euclidean geometry that tells us how circles and lines could cross; therefore, Hilbert needed an axiom stating this explicitly. However, we are allowing ourselves the use of algebra and thus can reduce this axiom down to a simple system of equations:

$$(x-h)^2+(y-k)^2=r^2, y=ax+c$$

   for the case of a line and a circle, and

$$(x-h_1)^2+(y-k_1)^2=r_1^2, (x-h_2)^2+(y-k_2)^2=r_2^2$$

   for the case of two circles. The first can have at most two solutions (none, one, or two), while the second system can have none, one, two, or infinitely many solutions (if $h_1=h_2$, $k_1=k_2$, and $r_1=r_2$).

5. We are all familiar with the fact that complex numbers can be represented on a coordinate plane along a real and imaginary axis. What fewer teachers are familiar with is the fact that multiplying complex numbers then results in certain types of transformations of the plan. In problem 5 we see that multiplication by i rotates the complex plane counterclockwise by $90°$, while multiplying by $-i$ rotates the plane clockwise $90°$

6. in problem 6 we see that multiplying by $1+i$ results in a counterclockwise rotation by $45°$ along with a scaling by a factor of $\sqrt{2}$

7. In problem 7 we see that multiplying by $\frac{3}{5}+\frac{4}{5}i$ results in an approximate rotation by $53.13°$ ($\tan^{-1}\left(\frac{4}{3}\right)$ to be exact) but there is do scaling.

8. and 9. These last five problems can be summed up by pointing out that when multiplying by a complex number $z=x+yi$ with a magnitude of $r=\sqrt{x^2+y^2}$ and an argument (angle with the x-axis) of $\theta=\tan^{-1}\left(\frac{y}{x}\right)$, then multiplying any complex number by z results in a rotation of θ and a scaling by a factor of r.

[1] the title of the paper is HILBERT'S AXIOM SYSTEM FOR PLANE GEOMETRY A SHORT INTRODUCTION. It can be found at `http://folk.uio.no/bjoernj/kurs/4510/hilberteng.pdf`