Math 5270
Transformational Geometry

Day 4

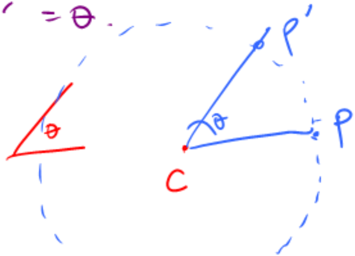
Summer 13

Rotations

The image of P under the rotation about C by angle θ is the point P′

Definition

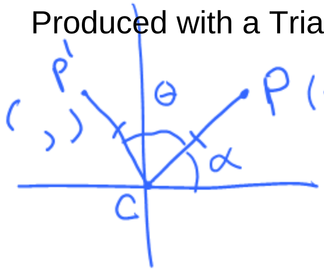
- using angle *theta* (θ) explicitly. Given a point P and a center C a rotation of P to P′ is such that $d(C,P) = d(C,P')$ and $\measuredangle PCP' = \theta$.



- using angle *theta* (θ) implicitly.

$(x, y) \longmapsto (x', y')$

what are these?



$P(x,y) = (r\cos\alpha, r\sin\alpha)$

$[r\cos(\theta+\alpha), r\sin(\theta+\alpha)]$

$r\cos(\theta+\alpha)$
$= r[\cos\theta\cos\alpha - \sin\theta\sin\alpha]$
$x(\cos\theta) - y(\sin\theta)$

$r\sin(\theta+\alpha)$
$= r[\sin\theta\cos\alpha + \cos\theta\sin\alpha]$
$x(\sin\theta) + y(\cos\theta)$

$(x,y) \longmapsto (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta)$

$(x,y) \longmapsto (cx - sy, sx + cy)$

for given $c, s \in \mathbb{R}$ $\quad c^2 + s^2 = 1$

we define $r_{c,s} : \mathbb{R}^2 \to \mathbb{R}^2$ as

$$r_{c,s}(x, y) = (cx - sy, \; sx + cy)$$

What does this do?

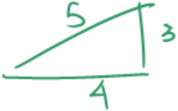
$(0,0) \longmapsto (0,0)$

$(1,0) \longmapsto (c,s)$

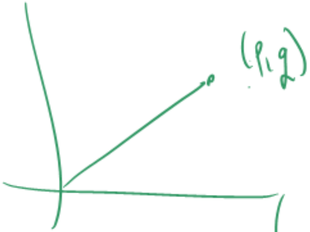
$(0,1) \longmapsto (-s,c)$

$$p(x,y) = (0.6x - 0.8y, \; 0.8x + 0.6y)$$

$$\left(\tfrac{3}{5}x - \tfrac{4}{5}y, \; \tfrac{4}{5}x + \tfrac{3}{5}y\right)$$



What if I have any numbers p and q?



$\sqrt{p^2+q^2}$ ← magnitude of (p,q)

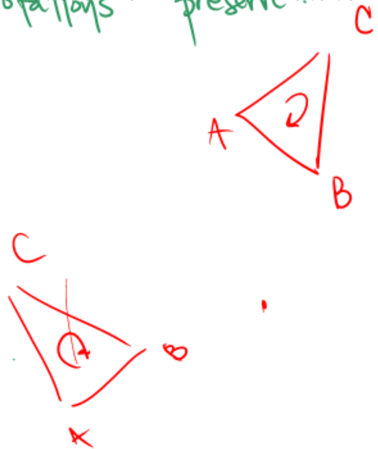
$\left(\frac{p}{\sqrt{p^2+q^2}}, \frac{q}{\sqrt{p^2+q^2}}\right)$ lives on unit circle

$\qquad\;\, \| \qquad\qquad\quad \|$

$(\cos\alpha \;, \sin\alpha)$

What else do rotations preserve?

- distance
- orientation
- angles
- collinearity
- parallels



Fixed points,fixed sets?

↳ center of rotation

circles centered at the center of rotation.

What happens when you compose two rotations?

① $r_{c,s}$ ($\to \alpha$) and $r_{a,b}$ ($\to \beta$) are both rotations about the origin.

$$r_{a,b} \circ r_{c,s}(x,y) = r_{a,b}\left(r_{c,s}(x,y)\right) =$$

$$= r_{a,b}(cx - sy,\ sx + cy) =$$

$$= \left(a(cx-sy) - b(sx+cy),\ b(cx-sy) + a(sx+cy)\right) =$$

$$= (acx - asy - bsx - bcy,\ bcx - bsy + asx + acy) =$$

$$= \left((ac-bs)x - (as+bc)y,\ (as+bc)x + (ac-bs)y\right)$$

$$= (px - qy,\ qx + py) \leftarrow \text{rotation by } \alpha + \beta$$

Is $p^2 + q^2 = 1$

$\hookrightarrow p = \cos(\alpha+\beta)$

$\hookrightarrow q = \sin(\alpha+\beta)$

$$(ac - bs)^2 + (as + bc)^2 =$$

$$= (ac)^2 - \cancel{2acbs} + (bs)^2 + (as)^2 + \cancel{2asbc} + (bc)^2 =$$

$$= a^2\underbrace{(c^2 + s^2)}_{=1} + b^2\underbrace{(c^2+s^2)}_{=1} = 1$$

$$\cos(\alpha+\beta) = p = ac - bs = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$\sin(\alpha+\beta) = q = as + bc = \sin\beta\cos\alpha + \sin\alpha\cos\beta$$

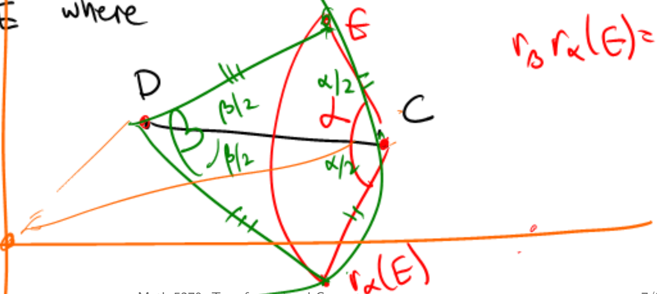
(2) What happens if you compose two rotations whose centers aren't eqeal?

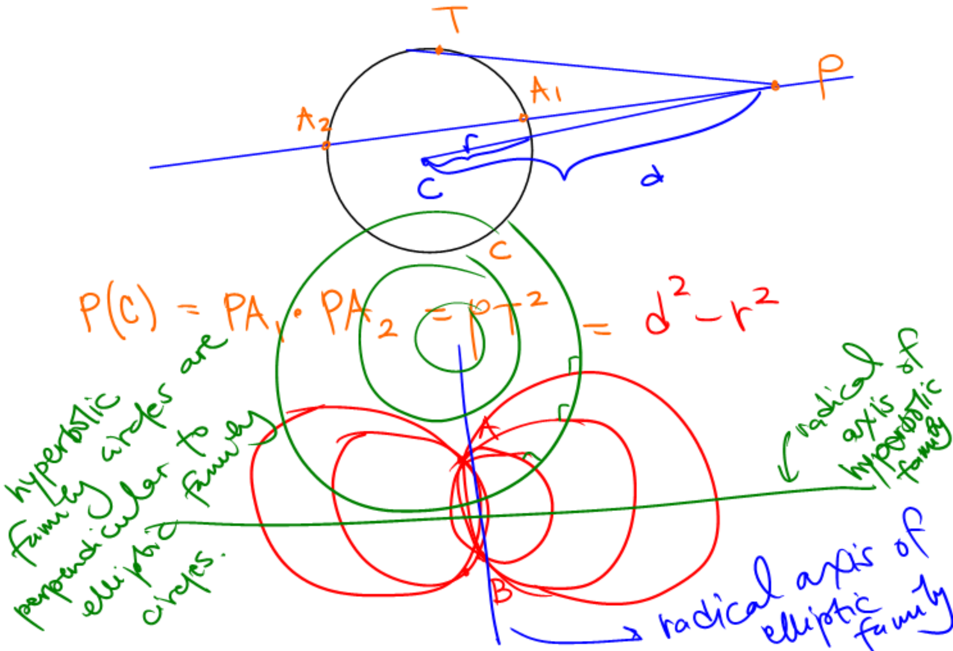
Conjecture: r_α: rotate around C by α, $r_{C,\alpha}$ ($\alpha+\beta<360°$)
r_β rotate around D by β, $r_{D,\beta}$

$r_\beta \circ r_\alpha$ = rotation by $\alpha+\beta$ about the point E where

$r_\beta r_\alpha(E)=E$

How do we get an equation for $r_\beta \circ r_\alpha$?





T
P
A_1
A_2
r
C
d
c
$P(C) = PA_1 \cdot PA_2 = PT^2 = d^2 - r^2$
hyperbolic family circles are perpendicular to elliptic circles.
family
radical axis of hyperbolic family
A
B
radical axis of elliptic family