

Math 431 Fall 2006 Solutions to Homework 1

1. Use truth tables to verify the following:

- (a) The negation of “if  $H$  then  $C$ ” is logically equivalent to “ $H$  and not  $C$ ”. (This corresponds to “Logic Rule 4”.)

$H$	$C$	if $H$ then $C$	not-(if $H$ then $C$ )	$H$ and not- $C$
$T$	$T$	$T$	<b>F</b>	<b>F</b>
$T$	$F$	$F$	<b>T</b>	<b>T</b>
$F$	$T$	$T$	<b>F</b>	<b>F</b>
$F$	$F$	$T$	<b>F</b>	<b>F</b>

The last two columns agree, so the last two statements are logically equivalent.

- (b) The statement “if  $P$  then  $Q$ ” is logically equivalent to its contrapositive: “if not- $Q$  then not- $P$ ”. (This corresponds to “Logic Rule 9(c)”.)

$P$	$Q$	if $P$ then $Q$	not- $Q$	not- $P$	if not- $Q$ then not- $P$
$T$	$T$	<b>T</b>	$F$	$F$	<b>T</b>
$T$	$F$	<b>F</b>	$T$	$F$	<b>F</b>
$F$	$T$	<b>T</b>	$F$	$T$	<b>T</b>
$F$	$F$	<b>T</b>	$T$	$T$	<b>T</b>

The two columns in bold agree, so the corresponding statements are logically equivalent.

2. Write English sentences, in language as simple as possible, giving the negation of each axiom of incidence. Avoid the word “unique” in negating the first axiom.

Negation of I-1: There exist two (distinct) points such that there is no line passing through both points or there are at least two lines passing through both points.

Negation of I-2: There exists a line that is incident with no more than one point.

Negation of I-3: For any three (distinct) points, there is a line passing through all three of them.

More succinctly: Any three points lie on a line.

**3.** In incidence geometry, for every line there is at least one point not lying on it.

*Proof.* Let  $l$  be any line. By I-3 there are three points not all lying on the same line. In particular, these three points do not all lie on  $l$ , meaning at least one of these points does not lie on  $l$ .  $\square$

**4.** In incidence geometry, for every point there is a line that does not pass through it.

*Proof.* Let  $P$  be any point. We wish to show that there is a line that does not pass through  $P$ . By I-3 there are three distinct points  $R, S, T$  that do not all lie on the same line. By I-1 there is a line  $l$  through  $R, S$ . There are two possibilities:  $l$  does not pass through  $P$  or  $l$  passes through  $P$ . If  $l$  does not pass through  $P$  then there is nothing more to prove; we have found our line.

Let us now consider the second possibility. Suppose that  $l$  passes through  $P$ . Since  $T, R, S$  do not all lie on the same line, and since we know that  $P, R, S$  all lie on  $l$ , it must be that  $P \neq T$ . I-1 gives us a line  $m$  passing through  $R, T$  and a line  $n$  passing through  $S, T$ . We claim that at least one of the lines  $m$  or  $n$  does not pass through  $P$ . Suppose for the sake of a contradiction that  $P$  lies on both  $m$  and  $n$ . Since  $T$  also lies on both of these lines, we are forced to conclude, by I-1, that  $m = n$ . Since  $R, T$  lie on  $m$  and  $S, T$  lie on  $n$ , it follows that  $R, S, T$  all lie on  $m$  (and  $n$ ), contradicting the fact that these three points do not all lie on the same line. Therefore, at least one of the two lines  $m$  or  $n$  does not pass through  $P$ .  $\square$