Section 6.1.

#19: Use the determinant to find out for which values of k the given matrix is invertible.

$$\begin{bmatrix} 1 & 1 & k \\ 1 & k & k \\ k & k & k \end{bmatrix}$$ is invertible iff its determinant $\neq 0$

$$\det\begin{bmatrix} 1 & 1 & k \\ 1 & k & k \\ k & k & k \end{bmatrix} \begin{matrix} 1 & 1 \\ 1 & k \\ k & k \end{matrix} = k^2+k^2+k^2-k^3-k^2-k = -k^3+2k^2-k = -k(k^2-2k+1) =$$
$$= -k(k-1)^2 \neq 0 \quad \text{iff} \quad k\neq 0 \text{ and } k\neq 1.$$

#33. Find the determinant of

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 8 & 7 & 6 & 5 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 7 & 5 \end{bmatrix}.$$ We can partition this matrix:

$$\det\left[\begin{array}{cc|cc} 1 & 2 & 3 & 4 \\ 8 & 7 & 6 & 5 \\ \hline 0 & 0 & 2 & 3 \\ 0 & 0 & 7 & 5 \end{array}\right] = \det\begin{bmatrix} 1 & 2 \\ 8 & 7 \end{bmatrix}\cdot\det\begin{bmatrix} 2 & 3 \\ 7 & 5 \end{bmatrix} = (7-16)\cdot(10-21) =$$
$$= -9\cdot(-11) = 99$$

#41.
$$\det\begin{bmatrix} 0 & 0 & 1 & 0 & 2 \\ 5 & 4 & 3 & 2 & 1 \\ 1 & 3 & 5 & 0 & 7 \\ 2 & 0 & 4 & 0 & 6 \\ 0 & 0 & 3 & 0 & 4 \end{bmatrix} = (-1)^{2+4}\cdot 2\cdot\det\begin{bmatrix} 0 & 0 & 1 & 2 \\ 1 & 3 & 5 & 7 \\ 2 & 0 & 4 & 6 \\ 0 & 0 & 3 & 4 \end{bmatrix} =$$
$$= 2\cdot(-1)^{2+2}\cdot 3\cdot\det\begin{bmatrix} 0 & 1 & 2 \\ 2 & 4 & 6 \\ 0 & 3 & 4 \end{bmatrix} = 6\cdot(-1)^{2+1}\cdot 2\cdot\det\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} =$$
$$= -12\cdot(4-6) = 24.$$

#45. If A is a 2×2 matrix what is the relationship between $\det A$ & $\det A^T$

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\qquad \det A = ad-bc$

$A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ $\qquad \det A^T = ad-bc = \det A.$

#46: If A is an invertible 2×2 matrix what is the relationship between $\det A$ & $\det A^{-1}$?

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ A is invertible $\Rightarrow$ not both a & c can be 0, so we assume $a \neq 0$.

$$\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & b/a & 1/a & 0 \\ c & d & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & b/a & 1/a & 0 \\ 0 & \frac{ad-bc}{a} & -\frac{c}{a} & 1 \end{array}\right] \sim$$

$$\sim \left[\begin{array}{cc|cc} 1 & b/a & 1/a & 0 \\ 0 & 1 & -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right] \sim \left[\begin{array}{cc|cc} & & \frac{d}{ad-bc} & -\frac{b}{ad-bc} \\ 0 & 1 & -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right] \Rightarrow$$

$$\Rightarrow A^{-1} = \begin{bmatrix} \frac{d}{ad-bc} & -\frac{b}{ad-bc} \\ -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix} \Rightarrow \det A^{-1} = \frac{d}{ad-bc}\cdot\frac{a}{ad-bc} - \frac{-b}{ad-bc}\cdot\frac{-c}{ad-bc}$$

$$= \frac{ad-bc}{(ad-bc)^2} = \frac{1}{ad-bc} = \frac{1}{\det A}.$$

Section 6.2:

#8: Use Gaussian elimination to find the determinant of:

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 2 \\ 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix} = B$$

switch rows 1 & 2
— " — 2 & 3
— " — 3 & 4
— " — 4 & 5

$\det B = (-1)^4 \det A \Rightarrow \det A = \det B = 2.$

#29: P_n is an $n\times n$ matrix whose all entries are 1 except for zeros directly below diagonal. Find $\det P_n$.

$$\det P_n = \det \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 & 1 \\ 0 & 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 0 & 1 & 1 & & 1 & 1 \\ 1 & 1 & 0 & 1 & \cdots & 1 & 1 \\ & & & & & & \\ 1 & 1 & 1 & 1 & & 0 & 1 \end{bmatrix} = (-1)^{1+1}\cdot 1\cdot \det \underbrace{\begin{bmatrix} 1 & 1 & \cdots & & 1 \\ 0 & 1 & & & 1 \\ 1 & 0 & \cdots & & 1 \\ \vdots & & & & \\ 1 & 1 & \cdot & 0 & 1 \end{bmatrix}}_{= P_{n-1}} +$$

$$+ (-1)^{3+1}\cdot 1\cdot \det \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ & & & & \\ 1 & 1 & \cdots & 0 & 1 \end{bmatrix} + \cdots + (-1)^{n+1}\cdot 1 \det \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 1 & \cdots & 1 \\ \vdots & & & & \\ 1 & 1 & 1 & \cdots 0 & 1 \end{bmatrix}$$

Note that every minor $(P_n)_{i1}$, $i=3,\ldots,n$ has two rows consisting of all 1s $\Rightarrow$ $\det(P_n)_{i1}=0$, $i=3,\ldots,n$, so we have

$$\det P_n = \det P_{n-1}, \quad \forall n.$$

$$\det P_2 = \det \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = 1 \quad \Rightarrow \quad \det P_n = 1, \quad \forall n.$$

#37. Consider an $n\times n$ matrix A such that both A & A^{-1} have integer entries. What are possible values for $\det A$?

If A has integer values then $\det A$ is an integer, too.
If A^{-1} —ll— $\det A^{-1}$ —ll—. Further

$$\underbrace{\det A}_{\in \mathbb{Z}} \cdot \underbrace{\det A^{-1}}_{\in \mathbb{Z}} = 1 \Rightarrow \det A = \pm 1$$

#46. Given some numbers a,b,c,d,e & f st $\det \begin{bmatrix} a & 1 & d \\ b & 1 & e \\ c & 1 & f \end{bmatrix} = 7$ & $\det \begin{bmatrix} a & 1 & d \\ b & 2 & e \\ c & 3 & f \end{bmatrix} =$

a) find

$$\det \begin{bmatrix} a & 3 & d \\ b & 3 & e \\ c & 3 & f \end{bmatrix} = 3\cdot \det \begin{bmatrix} a & 1 & d \\ b & 1 & e \\ c & 1 & f \end{bmatrix} = 3\cdot 7 = 21$$

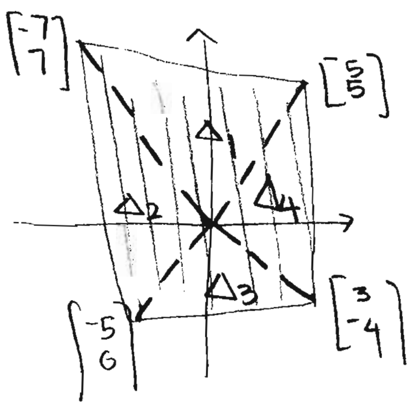
b) find

$$\det \begin{bmatrix} a & 3 & d \\ b & 5 & e \\ c & 7 & f \end{bmatrix} = \det \begin{bmatrix} a & 2\cdot 1+1 & d \\ b & 2\cdot 2+1 & e \\ c & 2\cdot 3+1 & f \end{bmatrix} = 2\det \begin{bmatrix} a & 1 & d \\ b & 2 & e \\ c & 3 & f \end{bmatrix} + \det \begin{bmatrix} a & 1 & d \\ b & 1 & e \\ c & 1 & f \end{bmatrix}$$

$$= 2\cdot 11 + 7 = 29$$

Section 3.6:

#7: Find the area of the following region:



$$A = \Delta_1 + \Delta_2 + \Delta_3 + \Delta_4 =$$

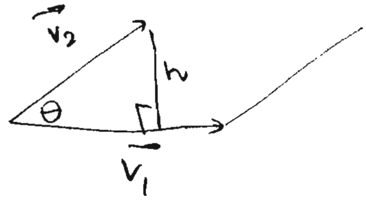
$$= \frac{1}{2}\left(\left|\det \begin{bmatrix} 5 & -7 \\ 5 & 7 \end{bmatrix}\right| + \left|\det \begin{bmatrix} -7 & -5 \\ 7 & 6 \end{bmatrix}\right| + \left|\det \begin{bmatrix} -5 & 3 \\ 6 & -4 \end{bmatrix}\right| + \left|\det \begin{bmatrix} 3 & 5 \\ -4 & 5 \end{bmatrix}\right| \right) =$$

$$= \frac{1}{2}\left(|35+35| + |-42+35| + |20-18| + |15+20| \right) =$$

$$= \frac{1}{2}(70+7+2+35) = \frac{114}{2} = 57$$

#9: Suppose two nonzero vectors $\vec{v}_1$ & $\vec{v}_2 \in \mathbb{R}^2$ enclose an angle θ ($0 \le \theta \le \pi$.). Explain why

$$|\det[\vec{v}_1\ \vec{v}_2]| = \|\vec{v}_1\| \cdot \|\vec{v}_2\| \cdot \sin\theta$$



$|\det[\vec{v}_1\ \vec{v}_2]|$ is the area of the parallelogra to the right

$$\sin\theta = \frac{h}{\|\vec{v}_2\|} \Rightarrow$$

$$\Rightarrow h = \|\vec{v}_2\| \cdot \sin\theta$$

$$|\det[\vec{v}_1\ \vec{v}_2]| = A = \|\vec{v}_1\| \cdot h = $$
$$= \|\vec{v}_1\| \cdot \|\vec{v}_2\| \cdot \sin\theta$$

#14: Find the 3-volume of the parallelopiped defined by vectors $\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\\1\end{bmatrix}$ & $\begin{bmatrix}1\\2\\3\\4\end{bmatrix}$. Let $A = \begin{bmatrix}1&1&1\\0&1&2\\0&1&3\\0&1&4\end{bmatrix}$.

$$\text{volume} = \sqrt{\det(A^TA)} = \left(\det\left(\begin{bmatrix}1&0&0&0\\1&1&1&1\\1&2&3&4\end{bmatrix}\begin{bmatrix}1&1&1\\0&1&2\\0&1&3\\0&1&4\end{bmatrix}\right)\right)^{1/2} =$$

$$= \left(\det\begin{bmatrix}1&1&1\\1&4&10\\1&10&30\end{bmatrix}\right)^{1/2} = \sqrt{120+10+10-4-100-30} = \sqrt{140-134} = \sqrt{6}$$

#24: Using Cramer's rule solve the system

$$\begin{aligned} 2x + 3y \quad &= 8 \\ 4y + 5z &= 3 \\ 6x \quad + 7z &= -1 \end{aligned}$$

$$A = \begin{bmatrix}2&3&0\\0&4&5\\6&0&7\end{bmatrix} \quad \text{\& } \det A = 56+90=146$$

$$x = \frac{\det\begin{bmatrix}8&3&0\\3&4&5\\-1&0&7\end{bmatrix}}{\det A} = \frac{32\cdot 7 - 15 - 9\cdot 7}{146} = \frac{146}{146} = 1$$

$$y = \frac{\det\begin{bmatrix}2&8&0\\0&3&5\\6&-1&7\end{bmatrix}}{\det A} = \frac{42+240+10}{146} = \frac{292}{146} = 2$$

$$z = \frac{\det\begin{bmatrix}2&3&8\\0&4&3\\6&0&-1\end{bmatrix}}{\det A} = \frac{-8+54-192}{146} = -1$$

$$\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}1\\2\\-1\end{bmatrix}.$$

\# 33. Find the classical adjoint of $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$.

$(\operatorname{adj} A)_{ij} = (-1)^{i+j} \det(A_{ji})$

$(\operatorname{adj} A)_{11} = (-1)^2 \det(A_{11}) = \det \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} = 24$

$(\operatorname{adj} A)_{12} = (-1)^3 \det A_{21} = -\det \begin{bmatrix} 0 & 0 & 0 \\ & \vdots & \\ & & \end{bmatrix} = 0$

$(\operatorname{adj} A)_{13} = (-1)^4 \det A_{31} = \det \begin{bmatrix} 0 & 0 & 0 \\ & \vdots & \\ & & \end{bmatrix} = 0$

$(\operatorname{adj} A)_{14} = (-1)^5 \det A_{41} = \det \begin{bmatrix} 0 & 0 & 0 \\ & \vdots & \\ & & \end{bmatrix} = 0$

$(\operatorname{adj} A)_{21} = (-1)^3 \det A_{12} = -\det \begin{bmatrix} 0 & 0 & 0 \\ & \cdots & \\ & & \end{bmatrix} = 0$

$(\operatorname{adj} A)_{22} = (-1)^4 \det A_{22} = \det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} = 12$

$(\operatorname{adj} A)_{23} = -\det A_{32} = -\det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ & & \end{bmatrix} = 0$

$(\operatorname{adj} A)_{24} = \det A_{42} = \det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ & \cdots & \end{bmatrix} = 0$

$(\operatorname{adj} A)_{31} = \det A_{13} = \det \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \\ & - & \end{bmatrix} = 0$

$(\operatorname{adj} A)_{32} = -\det A_{23} = -\det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ & - & \end{bmatrix} = 0$

$(\operatorname{adj} A)_{33} = \det A_{33} = \det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} = 8$

$(\operatorname{adj} A)_{41} = -\det A_{14} = -\det \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix} = 0$

$(\operatorname{adj} A)_{42} = \det A_{24} = \det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix} = 0$

$(\operatorname{adj} A)_{43} = -\det A_{34} = -\det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$

$(\operatorname{adj} A)_{44} = \det A_{44} = \det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = 6$

$$\operatorname{adj} A = \begin{bmatrix} 24 & 0 & 0 & 0 \\ 0 & 12 & 0 & 0 \\ 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix}$$

\# 36. For an invertible $n\times n$ matrix A what is $\operatorname{adj}(\operatorname{adj} A)$?

A is invertible then $A^{-1} = \frac{1}{\det A} \cdot \operatorname{adj} A \Rightarrow$

$\Rightarrow \operatorname{adj} A = \det A \cdot A^{-1}$ is itself an invertible matrix &

$$(\operatorname{adj} A)^{-1} = \frac{1}{\det(\operatorname{adj} A)} \cdot \operatorname{adj}(\operatorname{adj} A) \Rightarrow$$

$\operatorname{adj}(\operatorname{adj} A) = \det(\operatorname{adj} A) \cdot (\operatorname{adj} A)^{-1}$

Since $\operatorname{adj} A = \det A \cdot A^{-1}$ then $\det(\operatorname{adj} A) = (\det A)^n \cdot \det A^{-1} = (\det A)^{n-1}$

& $(\operatorname{adj} A)^{-1} = \frac{1}{\det A} \cdot A$

$$\Rightarrow \operatorname{adj}(\operatorname{adj} A) = (\det A)^{n-1} \cdot \frac{A}{\det A} = (\det A)^{n-2} \cdot A.$$