Section 3.4

#27 Find the matrix $\mathfrak{B}$ of $T(\vec{x}) = A\vec{x}$ with respect to $\mathfrak{B} = (\vec{v}_1, \ldots, \vec{v}_m)$

$$A = \begin{bmatrix} 4 & 2 & -4 \\ 2 & 1 & -2 \\ -4 & -2 & 4 \end{bmatrix}, \quad \vec{v}_1 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

$$A\vec{v}_1 = \begin{bmatrix} 4 & 2 & -4 \\ 2 & 1 & -2 \\ -4 & -2 & 4 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 18 \\ 9 \\ -18 \end{bmatrix} = 9\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} = 9\vec{v}_1$$

$$A\vec{v}_2 = \begin{bmatrix} 4 & 2 & -4 \\ 2 & 1 & -2 \\ -4 & -2 & 4 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0\cdot\vec{v}_1 + 0\cdot\vec{v}_2 + 0\cdot\vec{v}_3$$

$$A\vec{v_3} = \begin{bmatrix} 4 & 2 & -4 \\ 2 & 1 & -2 \\ -4 & -2 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \qquad B = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

#34. $\mathcal{B} = (\vec{v_1}, \vec{v_2}, \vec{v_3})$ basis of $\mathbb{R}^3$ consisting of perpendicular unit vectors. & $\vec{v_3} = \vec{v_1} \times \vec{v_2}$. Find the $\mathcal{B}$-matrix for

$$T(\vec{x}) = \vec{x} - 2(\vec{v_3} \cdot \vec{x}) \cdot \vec{v_3}$$

Solution:

$$T(\vec{v_1}) = \vec{v_1} - 2(\vec{v_3} \cdot \vec{v_1}) \cdot \vec{v_3} = \vec{v_1}$$
$$T(\vec{v_2}) = \vec{v_2} - 2(\vec{v_3} \cdot \vec{v_2}) \cdot \vec{v_3} = \vec{v_2} \qquad \Rightarrow \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$
$$T(\vec{v_3}) = \vec{v_3} - 2(\vec{v_3} \cdot \vec{v_3}) \cdot \vec{v_3} = -\vec{v_3}$$

$\Rightarrow$ T represents a reflection about the plane spanned by $\vec{v_1}$ & $\vec{v_2}$.

#39. Find $\mathcal{B}$ of $\mathbb{R}^3$ st $\mathcal{B}$-matrix of T is diagonal if T is a reflection about the line in $\mathbb{R}^3$ spanned by $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

Solution

We are looking for $\mathcal{B} = (\vec{v_1}, \vec{v_2}, \vec{v_3})$ so that

$$\left.\begin{array}{l} T(\vec{v_1}) = k_1 \vec{v_1} \\ T(\vec{v_2}) = k_2 \vec{v_2} \\ T(\vec{v_3}) = k_3 \vec{v_3} \end{array}\right. , \text{ because then } \quad \mathcal{B} = \begin{bmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{bmatrix}.$$

for $\vec{v_1}$ we can take $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, since $T\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ & $k_1 = 1$.

For $\vec{v_2}$ & $\vec{v_3}$ we can take vectors that are perpendicular to $\vec{v_1}$, since then $T(\vec{v_2}) = -\vec{v_2}$ & $T(\vec{v_3}) = -\vec{v_3}$, so $k_2 = k_3 = -1$.

We are looking for $\vec{v_2}$ & $\vec{v_3}$ st $\vec{v_3}\cdot\vec{v_1} = 0$ & $\vec{v_2}\cdot\vec{v_1} = 0$, i.e. we're looking for the kernel of $A = [1 \quad 2 \quad 3]$.

$\Rightarrow x_1 = -2x_2 - 3x_3$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow \vec{v_2} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \text{ \& } \vec{v_3} = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}.$$

#46. Consider the plane $x_1 + 2x_2 + x_3 = 0$. Find a basis $\mathcal{B}$ of this plane such that $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ for $\vec{x} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$.

Solution:

$V \ldots \quad x_1 + 2x_2 + x_3 = 0$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in V \iff x_1 = -2x_2 - x_3$$

We are looking for $\vec{v_1} = \begin{bmatrix} -2a-b \\ a \\ b \end{bmatrix}$ & $\vec{v_2} = \begin{bmatrix} -2c-d \\ c \\ d \end{bmatrix}$

st
$$\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = 2\begin{bmatrix} -2a-b \\ a \\ b \end{bmatrix} - \begin{bmatrix} -2c-d \\ c \\ d \end{bmatrix} = \begin{bmatrix} -4a-2b+2c+d \\ 2a-c \\ 2b-d \end{bmatrix} \Rightarrow$$

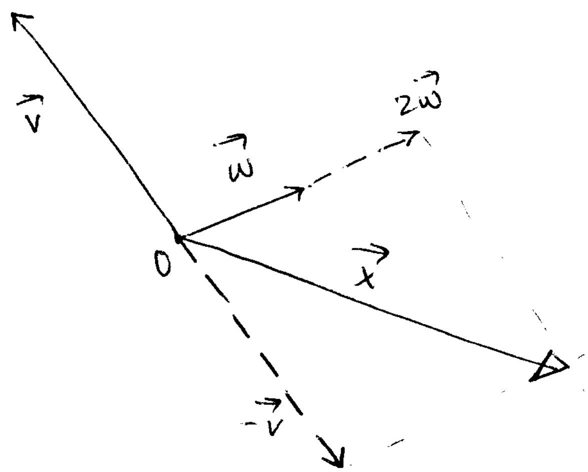
$$\Rightarrow \left[\begin{array}{cccc|c} 2 & 0 & -1 & 0 & -1 \\ 0 & 2 & 0 & -1 & 1 \\ -4 & -2 & 2 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{cccc|c} 2 & 0 & -1 & 0 & -1 \\ 0 & 2 & 0 & -1 & 1 \\ 0 & -2 & 0 & 1 & -1 \end{array}\right] \sim$$

$$\sim \left[\begin{array}{cccc|c} 2 & 0 & -1 & 0 & -1 \\ 0 & 2 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{ll} 2a - c = 1 & \Rightarrow c = 2a+1 \\ 2b - d = 1 & \Rightarrow d = 2b-1 \end{array}$$

$a = 1$ $\Rightarrow$ $c = 3$
$b = 0$ $\quad$ $d = -1$

$\rightarrow \vec{v}_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ & $\vec{v}_2 = \begin{bmatrix} -5 \\ 3 \\ -1 \end{bmatrix}$.

#48 Sketch vector $\vec{x}$ with $[\vec{x}]_\beta = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, where β is a basis consisting of $\vec{v}$ & $\vec{w}$:



#59. Is a matrix $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ similar to $\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$?

Solution:

Is there an invertible $S = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$ st

$$\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} x & y \\ z & w \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} 2x & 2y \\ 3z & 3w \end{bmatrix} = \begin{bmatrix} 2x & x+3y \\ 2z & z+3w \end{bmatrix} \Leftrightarrow$$

$2x = 2x$
$2y = x + 3y \Rightarrow x = -y$
$3z = 2z \Rightarrow z = 0$
$3w = z + 3w \quad \Rightarrow 3w = 3w$

$\Leftrightarrow S = \begin{bmatrix} -y & y \\ 0 & w \end{bmatrix}$ and if $y \neq 0 \neq w$ S is invertible, so the answer is YES.

Section 4.1.

\#5. Is $V = \{p(t) : p(-t) = -p(t), \forall t\}$ a subspace of P_2? If so, find its basis.

Solution

- Let $f, g \in V$ & $k, l \in \mathbb{R}$. Then

$$(kf+lg)(-t) = (kf)(-t) + (lg)(-t) = kf(-t) + lg(-t) = -kf(t) - lg(t) =$$
$$= -(kf+lg)(t) \Rightarrow kf + lg \in V \Rightarrow V \text{ is a subspace.}$$

- $f \in V$ & $f(t) = at^2 + bt + c$

$f(-t) = at^2 - bt + c$

$-f(t) = -at^2 - bt - c$

$f(-t) = -f(t) \Rightarrow$

$at^2 - bt + c = -at^2 - bt - c \Rightarrow$

$\Rightarrow 2at^2 + 2c = 0 \Rightarrow a = 0$ & $c = 0$

$\Rightarrow f(t) = bt$

$\Rightarrow$ Basis of V is $\{t\}$.

\#10: Do all 3×3 matrices A st $\begin{bmatrix}1\\2\\3\end{bmatrix}$ is in the kernel of A form a subspace of $M_{3,3}$?

Solution:

Yes. Let $A, B \in M_{3,3}$ st $A\begin{bmatrix}1\\2\\3\end{bmatrix} = 0$ & $B\begin{bmatrix}1\\2\\3\end{bmatrix} = 0$. Let $k, l \in \mathbb{R}$. Then

$$(kA + lB)\begin{bmatrix}1\\2\\3\end{bmatrix} = kA\begin{bmatrix}1\\2\\3\end{bmatrix} + lB\begin{bmatrix}1\\2\\3\end{bmatrix} = k\cdot A\begin{bmatrix}1\\2\\3\end{bmatrix} + l\cdot B\begin{bmatrix}1\\2\\3\end{bmatrix} = k\cdot\vec{0} + l\cdot\vec{0} = \vec{0} \Rightarrow$$

$$\Rightarrow \begin{bmatrix}1\\2\\3\end{bmatrix} \in \ker(kA + lB).$$

#12. V = set of all infinite sequences.
W = set of all arithmetic sequences $= (a, a+k, a+2k, \ldots)$ for some $a, k \in \mathbb{R}$

Is W a subspace of V?

Solution:

Let $(x_0, x_1, x_2, \ldots); (y_0, y_1, y_2, \ldots) \in W$ & $k, l \in \mathbb{R}$.

$(x_0, x_1, x_2, \ldots) \in W \Rightarrow x_n = x_0 + n \cdot p$, for some $p \in \mathbb{R}$

$(y_0, y_1, y_2, \ldots) \in W \Rightarrow y_n = y_0 + n \cdot r$, for some $r \in \mathbb{R}$

$$k(x_0, x_1, x_2, \ldots) + l(y_0, y_1, y_2, \ldots) = (kx_0 + ly_0, kx_1 + ly_1, \ldots, kx_n + ly_n, \ldots) =$$
$$= (kx_0 + ly_0, \ldots, k(x_0 + np) + l(y_0 + nr), \ldots) =$$
$$= (kx_0 + ly_0, \ldots, (kx_0 + ly_0) + n \cdot (kp + lr), \ldots) =$$

= arithmetic sequence with initial element $kx_0 + ly_0$ & additive constant $kp + lr$ $\in W$.

$\Rightarrow W$ is a subspace of V.

#31. Find a basis & dimension of $V = \left\{ S \in M_{2,2} : \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} S = S \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\}$

Solution:

$$S = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in V \iff \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} S = S \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \iff$$

$$\iff \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \iff \begin{bmatrix} c & d \\ a & b \end{bmatrix} = \begin{bmatrix} a & -b \\ c & -d \end{bmatrix} \iff$$

$$\iff \begin{array}{l} c = a \\ d = -b \end{array} \Rightarrow S = \begin{bmatrix} a & b \\ a & -b \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \Rightarrow$$

$\Rightarrow \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ & $\begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$ span V. Since they are lin. ind, they form a basis of V $\Rightarrow \dim V = 2$

#35. $V = \{A \in M_{33} : AB = BA\}$; $B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

Find basis of V & its dimension

Solution:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \in V$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} 2a_{11} & 3a_{12} & 4a_{13} \\ 2a_{21} & 3a_{22} & 4a_{23} \\ 2a_{31} & 3a_{32} & 4a_{33} \end{bmatrix} = \begin{bmatrix} 2a_{11} & 2a_{12} & 2a_{13} \\ 3a_{21} & 3a_{22} & 3a_{23} \\ 4a_{31} & 4a_{32} & 4a_{33} \end{bmatrix}$$

$3a_{12} = 2a_{12} \Rightarrow a_{12} = 0$, Similarly for all off-diagonal entries

$$\Rightarrow A = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix} = a_{11}\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{A_1} + a_{22}\underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{A_2} + a_{33}\underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{A_3}$$

$\Rightarrow A_1, A_2, A_3$ span V and are lin independent $\Rightarrow (A_1, A_2, A_3)$ is a basis $\rightarrow \dim V = 3$.

#37: $V = \{A \in M_{33} : AB = BA\}$ where $B = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$

Basis of V & dim of V?

Solution:

as above we get

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \in V \iff \begin{bmatrix} aa_{11} & ba_{12} & ca_{13} \\ aa_{21} & ba_{22} & ca_{23} \\ aa_{31} & ba_{32} & ca_{33} \end{bmatrix} = \begin{bmatrix} aa_{11} & aa_{12} & aa_{13} \\ ba_{21} & ba_{22} & ba_{23} \\ ca_{31} & ca_{32} & ca_{33} \end{bmatrix}$$

- a, b, c are all distinct

then we are in the same situation as above and vectors

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ is a basis}$$

and $\underline{\dim V = 3}$

- $a \neq b = c$

$$\begin{bmatrix} aa_{11} & ba_{12} & ba_{13} \\ aa_{21} & ba_{22} & ba_{23} \\ aa_{31} & ba_{32} & ba_{33} \end{bmatrix} = \begin{bmatrix} aa_{11} & aa_{12} & aa_{13} \\ ba_{21} & ba_{22} & ba_{23} \\ ba_{31} & ba_{32} & ba_{33} \end{bmatrix} \Rightarrow \quad a_{12} = a_{13} = a_{21} = a_{31} = 0$$

$$\Rightarrow \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{bmatrix} = a_{11}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + a_{22}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + a_{23}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} + a_{32}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} + a_{33}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow \underline{\dim V = 5}$

- $a = b = c \Rightarrow B = a \cdot I \quad AB = A \cdot (aI) = aA \cdot I = a\,IA = BA$, for all 3×3 matrices $\rightarrow \underline{\dim V = 9}$

Section 4.2.

#7 Is $T: M_{2,2} \to M_{2,2}$ defined by $T(M) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} M$ a linear transf? Is it an isomorphism?

Solution

$$T(M+N) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}(M+N) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}M + \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}N = T(M) + T(N)$$

$$T(kM) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}kM = k \cdot \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}M = k \cdot T(M)$$

$\Rightarrow T$ is linear.

T is an isomorphism if $T(M)=N$ has a unique solution $\forall N \in M_{2,2}$.

$T(M)=N \iff \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} M = N$ has unique solution if

$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{-1}$ exists, because then $M = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{-1} \cdot N$ is the unique solution.

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -2 & -3 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & -2 & -3 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & 1 & 3/2 & -1/2 \end{array}\right]$$

$\Rightarrow$ T is an isomorphism.

#12. Same Q for $T: \mathbb{R} \to M_{2,2}$, $T(c) = c \cdot \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$

Solution

$k \in \mathbb{R}$ $\quad T(kc) = kc \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} = k \cdot T(c)$

$c_1, c_2 \in \mathbb{R}$ $\quad T(c_1+c_2) = (c_1+c_2)\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} = c_1 \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} + c_2 \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} = T(c_1) + T(c_2)$

$\Rightarrow$ T is linear, but it can't be an isomorphism:

$\dim \operatorname{Im} T = 1$, while $\dim M_{2,2} = 4$.

#24: $T: P_2 \to P_2$, $T(f(t)) = f''(t) \cdot f(t)$

Solution:

$f, g \in P_2$ $\quad T((f+g)(t)) = (f+g)''(t) \cdot (f+g)(t) =$

$= (f''(t) + g''(t)) \cdot (f(t) + g(t)) =$

$= f''(t) \cdot f(t) + f''(t) \cdot g(t) + g''(t) f(t) + g''(t) \cdot g(t)$

$\neq f''(t) f(t) + g''(t) g(t) = T(f(t)) + T(g(t))$

$\Rightarrow$ T is not linear.

#30. $T: P_2 \to P_2$ $\quad T(f(t)) = t\cdot(f'(t))$

Solution:

$f, g \in P_2$ $\quad T((f+g)(t)) = t\cdot(f+g)'(t) = t\cdot(f'(t)+g'(t)) =$
$= tf'(t) + tg'(t) = T(f(t)) + T(g(t))$

$k \in \mathbb{R}, f \in P_2$ $\quad T((kf)(t)) = t\cdot(kf)'(t) = k\cdot tf'(t) = kT(f(t))$

$\Rightarrow T$ is linear, but it is not an isomorphism, because $f(t) = c$, $c \neq 0$, is not in the image of T.

#36. $T: P \to V$, P space of all polynomials, V space of all infinite sequences

$T(f(t)) = (f(0), f(1), f(2), f(3), \ldots)$

Solution:

$f, g \in P$ $\quad T((f+g)(t)) = ((f+g)(0), (f+g)(1), \ldots) =$
$= (f(0)+g(0), f(1)+g(1), \ldots) =$
$= (f(0), f(1), \ldots) + (g(0), g(1), \ldots) = T(f(t)) + T(g(t))$

$f \in P, k \in \mathbb{R}$ $\quad T((kf)(t)) = ((kf)(0), (kf)(1), \ldots) = (kf(0), kf(1), \ldots) =$
$= k(f(0), f(1), \ldots) = k\cdot T(f(t))$

$\Rightarrow T$ is linear

Consider the sequence $(7, 0, 0, 0, \ldots) \in V$ whose first component is 7, and all others are 0. If $T(f(t)) = (7, 0, 0, \ldots)$, then $f(n) = 0$, $\forall n \geq 1$, so f has infinitely many zeros, which can be true only if f is a zero polynomial, but in that case $f(0) = 0 \neq 7$, so $(7, 0, 0, 0, \ldots) \notin \operatorname{Im} T \Rightarrow V \neq \operatorname{Im} T \Rightarrow T$ is not an isomorphism

#40 $T: C^\infty \to C^\infty$, $T(f) = f'' + 2f' + f$

T is linear

$T(f+g) = (f+g)'' + 2(f+g)' + f+g = f'' + g'' + 2f' + 2g' + f + g = T(f) + T(g)$

$T(kf) = (kf)'' + 2(kf') + (kf) = T(f)$

$= kf'' + 2kf' + kf = kT(f)$

Let's try to see if there is anything in the kernel of T.

$f(x) = e^{kx}$; $f'(x) = k \cdot e^{kx}$; $f''(x) = k^2 e^{kx}$

$T(e^{kx}) = k^2 e^{kx} + 2k e^{kx} + e^{kx} = (k^2 + 2k + 1) e^{kx} = 0$, $\forall x \iff$

$k^2 + 2k + 1 = (k+1)^2 = 0 \iff k = -1$

$\Rightarrow f(x) = e^{-x} \in \ker T \Rightarrow T$ is not an isomorphism.

#56 Find image, rank, kernel & nullity of T from #30.

Solution

$T: P_2 \to P_2$, $\dim P_2 = 3$ basis $= (1, t, t^2)$.

$T(f(t)) = t \cdot f'(t)$

$f \in \ker T \iff T(f(t)) = 0 \iff t \cdot f'(t) = 0 \iff f'(t) = 0 \iff f(t) = a$, $a \in \mathbb{R} \Rightarrow f(t) = a \cdot 1 \Rightarrow \ker T = \text{Span}(1) \Rightarrow \text{nullity}(T) = 1$

$f(t) = a + bt + ct^2 \Rightarrow f'(t) = b + 2ct \Rightarrow t \cdot f'(t) = bt + 2ct^2 = b \cdot t + c \cdot 2t^2$

$\Rightarrow \operatorname{Im} T$ is spanned by $\{t, 2t^2\}$ (which is a basis

$\Rightarrow \text{rank}(T) = 2$.

#64 Define an isomorphism from P_2 to $M_{2,2}$, if you can.

Can't be done ; $\dim P_2 = 3$ & $\dim M_{2,2} = 4$.

Except that wasn't a problem. Turn to the last page!

Define an isomorphism from P_3 to $M_{2,2}$.

$T: P_3 \to M_{2,2}$ $\quad T(a+bt+ct^2+dt^3) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Consider

$$\begin{tikzcd} P_3 \arrow[rr, "T"] \arrow[dr, "\text{coordinate map}"', "\mathcal{C}"] & & M_{2,2} \arrow[dl, "\text{coordinate map}", "\mathcal{B}"'] \\ & \mathbb{R}^4 \arrow[ur, dashed, "\mathcal{B}^{-1}"] & \end{tikzcd}$$

, We know $\mathcal{C}$ & $\mathcal{B}$ are isomorphisms

Then $\mathcal{C} = \mathcal{B} \circ T$, but $\mathcal{B}$ is invertible, so

$T = \mathcal{B}^{-1} \circ \mathcal{C}$ is a composition of two isomorphisms is an isomorphism.